

Neutrino electromagnetic interactions: a window to new physics

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We review the theory and phenomenology of neutrino electromagnetic interactions, which give us powerful tools to probe the physics beyond the Standard Model. After a derivation of the general structure of the electromagnetic interactions of Dirac and Majorana neutrinos in the one-photon approximation, we discuss the effects of neutrino electromagnetic interactions in terrestrial experiments and in astrophysical environments. We present the experimental bounds on neutrino electromagnetic properties and we confront them with the predictions of theories beyond the Standard Model.

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I. INTRODUCTION

The theoretical and experimental investigation of neutrino properties and interactions is one of the most active fields of research in current high-energy physics. It brings us precious information on the physics of the Standard Model and provides a powerful window on the physics beyond the Standard Model.

The possibility that a neutrino has a magnetic moment was considered by Pauli in his famous 1930 letter addressed to “Dear Radioactive Ladies and Gentlemen” (see Pauli (1991)), in which he proposed the existence of the neutrino and he supposed that its mass could be of the same order of magnitude as the electron mass. Neutrinos remained elusive until the detection of reactor neutrinos by Reines and Cowan around 1956 (Reines *et al.*, 1960). However, there was no sign of a neutrino mass and, after the discovery of parity violation in 1957, Landau (1957); Lee and Yang (1957); Salam (1957) proposed the two-component theory of massless neutrinos, in which a neutrino is described by a Weyl spinor and there are only left-handed neutrinos and right-handed antineutrinos. It was however clear (Radicati and Tuschek, 1957; Case, 1957; McLennan, 1957) that the two-component theory of a massless neutrino is equivalent to the Majorana theory in the limit of zero neutrino mass.

The two-component theory of massless neutrinos was later incorporated in the Standard Model of Glashow (1961); Weinberg (1967); Salam (1969), in which neutrinos are massless and have only weak interactions. In the Standard Model Majorana neutrino masses are forbidden by the $SU(2)_L \times U(1)_Y$ symmetry.

Since the epochal measurement of the up-down asymmetry of high-energy events generated by atmospheric muon neutrinos in the Super-Kamiokande experiment (Fukuda *et al.*, 1998), we know that the flavor of neutrinos oscillates with distance. Since this phenomenon of neutrino oscillations (Pontecorvo, 1957, 1958; Maki *et al.*, 1962; Pontecorvo, 1968) is generated by neutrino masses (see Giunti and Kim (2007); Bilenky (2010); Xing and Zhou (2011); Gonzalez-Garcia *et al.* (2012); Bellini *et al.* (1960)), the Standard Model must be extended to account for the neutrino masses.

There are many possible extensions of the Standard Model which predict different properties for the neutrinos (see Mohapatra and Pal (2004); Xing and Zhou (2011)). Among them, most important is their fundamental Dirac or Majorana character. In many extensions of the Standard Model neutrinos acquire also electromagnetic properties through quantum loops effects which allow direct interactions of neutrinos with electromagnetic fields and electromagnetic interactions of neutrinos with charged particles.

Hence, the theoretical and experimental study of neutrino electromagnetic interactions is a powerful tool in the search for the fundamental theory beyond the Stan-

dard Model. Moreover, the electromagnetic interactions of neutrinos can generate important effects, especially in astrophysical environments, where neutrinos propagate over long distances in magnetic fields in vacuum and in matter.

Unfortunately, in spite of many efforts in the search of neutrino electromagnetic interactions, up to now there is no positive experimental indication in favor of their existence. However, the existence of neutrino masses and mixing implies that non-trivial neutrino electromagnetic properties are plausible and experimentalists and theorists are eagerly looking for them.

The structure of this review is as follows. In Section II we summarize the basic theory of neutrinos masses and mixing and the phenomenology of neutrino oscillations, which are important for the following discussion of theoretical models and for understanding the connection between neutrino masses and mixing and neutrino electromagnetic properties. In Section III we derive the general form of the electromagnetic interactions of Dirac and Majorana neutrinos in the one-photon approximation, which are expressed in terms of electromagnetic form factors. We also discuss the derivation of the form factors in gauge models. In Section IV we discuss the phenomenology of the neutrino magnetic and electric dipole moments in laboratory experiments. These are the most studied electromagnetic properties of neutrinos, both experimentally and theoretically. In Section V we discuss neutrino radiative decay in vacuum and in matter and related processes which are induced by the neutrino magnetic and electric dipole moments. These processes could have observable effects in astrophysical environments and could be detected on Earth by astronomical photon detectors. In Section VI we discuss some important effects due to the interaction of neutrino magnetic moments with classical electromagnetic fields. In particular, we derive the effective potential in a magnetic field and we discuss the corresponding spin and spin-flavor transitions in astrophysical environments. In Section VII we review the theory and experimental constraints on the neutrino electric charge (millicharge), the charge radius and the anapole moment. In conclusion, in Section VIII we summarize the status of our knowledge of neutrino electromagnetic properties and we discuss the prospects for future research.

Let us also remind that neutrino electromagnetic properties and interactions are discussed in the books by Bahcall (1989); Boehm and Vogel (1992); Kim and Pevsner (1993); Raffelt (1996); Fukugita and Yanagida (2003); Zuber (2003); Mohapatra and Pal (2004); Xing and Zhou (2011); Barger *et al.* (2012); Lesgourgues *et al.* (2013), and in the previous reviews by Dolgov and Zeldovich (1981); Raffelt (1990b); Salati (1994); Raffelt (1999a,b, 2000); Pulido (1992a); Dolgov (2002); Nowakowski *et al.* (2005); Wong and Li (2005); Studenikin (2009); Giunti and Studenikin (2009); Brogini *et al.* (2012).

II. NEUTRINO MASSES AND MIXING

In the Standard Model of electroweak interactions (Glashow, 1961; Weinberg, 1967; Salam, 1969), neutrinos are described by two-component massless left-handed Weyl spinors (see Giunti and Kim (2007)). The masslessness of neutrinos is due to the absence of right-handed neutrino fields, without which it is not possible to have Dirac mass terms, and to the absence of Higgs triplets, without which it is not possible to have Majorana mass terms. In the following we consider the extension of the Standard Model with the introduction of three right-handed neutrinos. We will see that this seemingly innocent addition has the very powerful effect of introducing not only Dirac mass terms, but also Majorana mass terms for the right-handed neutrinos, which can induce Majorana masses for the observable light neutrinos through the see-saw mechanism.

Table I shows the values of the weak isospin, hypercharge, and electric charge of the lepton and Higgs doublets and singlets in the extended Standard Model under consideration. For simplicity, we work in the flavor basis in which the mass matrix of the charged leptons is diagonal. Hence, e , μ , τ are the physical charged leptons with definite masses.

In Subsections II.A and II.B we review the basic theory of neutrinos masses and mixing for Dirac and Majorana neutrinos, respectively. In Subsection II.C we present the standard framework of three-neutrino mixing and in Subsection II.D we review neutrino oscillations in vacuum and in matter. In Subsection II.E we review the current phenomenological status of three-neutrino mixing.

A. Dirac neutrinos

The fields in Tab. I allow us to construct the Yukawa Lagrangian term

$$\mathcal{L}_Y = - \sum_{\alpha, \beta=e, \mu, \tau} Y_{\alpha\beta} \overline{L_{\alpha L}} \tilde{\Phi} \nu_{\beta R} + \text{H.c.}, \quad (2.1)$$

where Y is a matrix of Yukawa couplings and $\tilde{\Phi} \equiv i\sigma_2 \Phi^*$. In the Standard Model, a nonzero vacuum expectation value of the Higgs doublet,

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad (2.2)$$

induces the spontaneous symmetry breaking of the Standard Model symmetries $\text{SU}(2)_L \times \text{U}(1)_Y \rightarrow \text{U}(1)_Q$. In the unitary gauge, the Higgs doublet is given by

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H \end{pmatrix}, \quad (2.3)$$

where H is the physical Higgs field. From the Yukawa Lagrangian term in Eq. (2.1), we obtain the neutrino Dirac mass term

$$\mathcal{L}_D = - \sum_{\alpha, \beta=e, \mu, \tau} \overline{\nu_{\alpha L}} M_{\alpha\beta}^D \nu_{\beta R} + \text{H.c.}, \quad (2.4)$$

with the complex 3×3 Dirac mass matrix

$$M^D = \frac{v}{\sqrt{2}} Y. \quad (2.5)$$

If the total lepton number is conserved, \mathcal{L}_D is the only neutrino mass term and the three massive neutrinos obtained through the diagonalization of \mathcal{L}_D are Dirac particles. The diagonalization of \mathcal{L}_D is achieved through the transformations

$$\nu_{\alpha L} = \sum_{k=1}^3 U_{\alpha k} \nu_{kL}, \quad (2.6)$$

$$\nu_{\beta R} = \sum_{k=1}^3 V_{\beta k} \nu_{kR}, \quad (2.7)$$

with unitary 3×3 matrices U and V such that

$$(U^\dagger M^D V)_{kj} = m_k \delta_{kj}, \quad (2.8)$$

with real and positive masses m_k (see Bilenky and Petcov (1987); Giunti and Kim (2007)). The resulting diagonal Dirac mass term is

$$\mathcal{L}_D = - \sum_{k=1}^3 m_k \overline{\nu_{kL}} \nu_{kR} + \text{H.c.} = - \sum_{k=1}^3 m_k \overline{\nu_k} \nu_k, \quad (2.9)$$

with the Dirac fields of massive neutrinos

$$\nu_k = \nu_{kL} + \nu_{kR}. \quad (2.10)$$

B. Majorana neutrinos

In the above derivation of Dirac neutrino masses we have assumed that the total lepton number is conserved. However, since there is not any compelling argument which imposes the conservation of the total lepton number, it is plausible that the right-handed singlet neutrinos have the Majorana mass term

$$\mathcal{L}_R = \frac{1}{2} \sum_{\alpha, \beta=e, \mu, \tau} \nu_{\alpha R}^T \mathcal{C}^\dagger M_{\alpha\beta}^R \nu_{\beta R} + \text{H.c.}, \quad (2.11)$$

which violates the total lepton number by two units. In Eq. (2.11), \mathcal{C} is the charge-conjugation matrix defined by Eqs. (A32)–(A34) and the mass matrix M^R is complex and symmetric.

The Majorana mass term in Eq. (2.11) is allowed by the symmetries of the Standard Model, since right-handed

$(\alpha = e, \mu, \tau)$		I	I_3	Y	Q
left-handed lepton doublets	$L_{\alpha L} \equiv \begin{pmatrix} \nu_{\alpha L} \\ \alpha_L \end{pmatrix}$	1/2	1/2 -1/2	-1	0 -1
right-handed charged-lepton singlets	α_R	0	0	-2	-1
right-handed neutrino singlets	$\nu_{\alpha R}$	0	0	0	0
Higgs doublet	$\Phi \equiv \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	1/2	1/2 -1/2	+1	1 0

TABLE I Eigenvalues of the weak isospin I , of its third component I_3 , of the hypercharge Y , and of the charge $Q = I_3 + Y/2$ of the lepton and Higgs doublets and singlets in the extension of the Standard Model with the introduction of right-handed neutrinos.

neutrino fields are invariant. On the other hand, an analogous Majorana mass term of the left-handed neutrinos,

$$\mathcal{L}_L = \frac{1}{2} \sum_{\alpha, \beta=e, \mu, \tau} \nu_{\alpha L}^T \mathcal{C}^\dagger M_{\alpha\beta}^L \nu_{\beta L} + \text{H.c.}, \quad (2.12)$$

is forbidden, since it has $I_3 = 1$ and $Y = -2$, as one can find easily using Tab. I. There is no Higgs triplet in the Standard Model to compensate these quantum numbers.

In the extension of the Standard Model with the introduction of right-handed neutrinos, the neutrino masses and mixing are given by the Dirac–Majorana mass term

$$\mathcal{L}_{D+M} = \mathcal{L}_D + \mathcal{L}_R. \quad (2.13)$$

The neutrino fields with definite masses are obtained through the diagonalization of \mathcal{L}_{D+M} . It is convenient to define the vector N_L of six left-handed fields

$$N_L^T \equiv (\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}, \nu_{eR}^c, \nu_{\mu R}^c, \nu_{\tau R}^c), \quad (2.14)$$

with the charge-conjugated fields

$$\nu_{\alpha R}^c = \mathcal{C} \overline{\nu_{\alpha R}}^T. \quad (2.15)$$

The Dirac–Majorana mass term in Eq. (2.13) can be written in the compact form

$$\mathcal{L}_{D+M} = \frac{1}{2} N_L^T \mathcal{C}^\dagger M^{D+M} N_L + \text{H.c.}, \quad (2.16)$$

with the 6×6 symmetric mass matrix

$$M^{D+M} \equiv \begin{pmatrix} 0 & M^{D^T} \\ M^D & M^R \end{pmatrix}. \quad (2.17)$$

The order of magnitude of the elements of the Dirac mass matrix M^D in Eq. (2.5) is smaller than $v \sim 10^2$ GeV, since the Dirac mass term (2.4) is forbidden by the symmetries of the Standard Model and can be generated only as a consequence of symmetry breaking below the electroweak scale v . On the other hand, since the Majorana mass term in Eq. (2.11) is a Standard Model singlet, the elements of the Majorana mass matrix M_R are

not related to the electroweak scale. It is plausible that the Majorana mass term \mathcal{L}_R is generated by new physics beyond the Standard Model and the right-handed chiral neutrino fields $\nu_{\alpha R}$ belong to nontrivial multiplets of the symmetries of the high-energy theory. The corresponding order of magnitude of the elements of the mass matrix M_R is given by the symmetry-breaking scale of the high-energy physics beyond the Standard Model, which may be as large as the grand unification scale, of the order of 10^{14} – 10^{16} GeV. In this case, the mass matrix can be diagonalized by blocks, up to corrections of the order $\epsilon = (M^R)^{-1} M^D$:

$$W^T M^{D+M} W \simeq \begin{pmatrix} M_l^M & 0 \\ 0 & M_h^M \end{pmatrix}, \quad (2.18)$$

with

$$W \simeq 1 - \frac{1}{2} \begin{pmatrix} \epsilon^\dagger \epsilon & 2\epsilon^\dagger \\ -2\epsilon & \epsilon \epsilon^\dagger \end{pmatrix}. \quad (2.19)$$

The light symmetric 3×3 Majorana mass matrix M_l^M and the heavy symmetric 3×3 Majorana mass matrix M_h^M are given by

$$M_l^M \simeq -M^{D^T} (M^R)^{-1} M^D, \quad M_h^M \simeq M^R. \quad (2.20)$$

There are three heavy masses given by the eigenvalues of M_h^M and three light masses given by the eigenvalues of M_l^M , whose elements are suppressed with respect to the elements of the Dirac mass matrix M^D by the very small matrix factor $M^{D^T} (M^R)^{-1}$. This is the celebrated *see-saw mechanism* (Minkowski, 1977; Gell-Mann *et al.*, 1979; Yanagida, 1979; Mohapatra and Senjanovic, 1980), which explains naturally the smallness of light neutrino masses. Notice, however, that the values of the light neutrino masses and their relative sizes can vary over wide ranges, depending on the specific values of the elements of M^D and M^R .

Since the off-diagonal block elements of W are very small, the three flavor neutrinos are mainly composed by

the three light neutrinos. Therefore, the see-saw mechanism implies the effective low-energy Majorana mass term

$$\mathcal{L}_M^{\text{eff}} = \frac{1}{2} \sum_{\alpha, \beta=e, \mu, \tau} \nu_{\alpha L}^T \mathcal{C}^\dagger (M_1^M)_{\alpha\beta} \nu_{\beta L} + \text{H.c.}, \quad (2.21)$$

which involves only the three active left-handed flavor neutrino fields. The symmetric 3×3 Majorana mass matrix M_1^M is diagonalized by the transformation in Eq. (2.6) with a 3×3 unitary mixing matrix U such that

$$(U^T M_1^M U)_{kj} = m_k \delta_{kj}, \quad (2.22)$$

with real and positive masses m_k (see Bilenky and Petcov (1987); Giunti and Kim (2007)). In this way, the effective Majorana mass term in Eq. (2.21) can be written in terms of the massive fields as

$$\begin{aligned} \mathcal{L}_M^{\text{eff}} &= \frac{1}{2} \sum_{k=1}^3 m_k \nu_{kL}^T \mathcal{C}^\dagger \nu_{kL} + \text{H.c.} \\ &= \frac{1}{2} \sum_{k=1}^3 m_k \nu_k^T \mathcal{C}^\dagger \nu_k, \end{aligned} \quad (2.23)$$

with the massive Majorana fields

$$\nu_k = \nu_{kL} + \mathcal{C} \overline{\nu_{kL}}^T, \quad (2.24)$$

which satisfy the Majorana constraint

$$\nu_k = \nu_k^c = \mathcal{C} \overline{\nu_k}^T. \quad (2.25)$$

Hence, a general result of the see-saw mechanism is an effective low-energy mixing of three massive Majorana neutrinos.

C. Three-neutrino mixing

In the previous two Sections we have seen that an effective mixing of three light neutrinos is obtained in the

Dirac case assuming the conservation of the total lepton number and in the Majorana case through the see-saw mechanism. In both cases the mixing relation between the three left-handed flavor neutrino fields $\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}$ which partake in weak interactions and the three left-handed massive neutrino fields $\nu_{1L}, \nu_{2L}, \nu_{3L}$ is given by Eq. (2.6), which depends on a unitary 3×3 mixing matrix U .

The mixing matrix U is observable through its effects in charged-current weak interaction processes in which leptons are described by the current

$$j_{CC}^\rho = 2 \sum_{\alpha=e, \mu, \tau} \overline{\nu_{\alpha L}} \gamma^\rho \alpha_L = 2 \sum_{\alpha=e, \mu, \tau} \sum_{k=1}^3 U_{\alpha k}^* \overline{\nu_{kL}} \gamma^\rho \alpha_L. \quad (2.26)$$

A unitary 3×3 matrix can be parameterized in terms of 3 mixing angles and 6 phases. However, in the mixing matrix 3 phases are unphysical, because they can be eliminated by rephasing the three charged lepton fields in j_{CC}^ρ . In the case of Majorana massive neutrinos, no additional phase can be eliminated, because the Majorana mass term in Eq. (2.23) is not invariant under rephasing of ν_{kL} . On the other hand, in the case of Dirac massive neutrinos, two additional phases can be eliminated by rephasing the massive neutrino fields. Hence, the mixing matrix has 3 physical phases in the case of Majorana massive neutrinos or 1 physical phase in the case of Dirac massive neutrinos. In general, in the case of Majorana massive neutrinos U can be written as

$$U = U^D D^M, \quad (2.27)$$

where U^D is a Dirac unitary mixing matrix which can be parameterized in terms of three mixing angles and one physical phase, called *Dirac phase*, and D^M is a diagonal unitary matrix with two physical phases, usually called *Majorana phases*. In the case of Dirac neutrinos $U = U^D$.

The standard parameterization of U^D is

$$U^D = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}, \quad (2.28)$$

where $c_{ab} \equiv \cos \vartheta_{ab}$ and $s_{ab} \equiv \sin \vartheta_{ab}$. $\vartheta_{12}, \vartheta_{13}, \vartheta_{23}$ are the three mixing angles ($0 \leq \vartheta_{ab} \leq \pi/2$) and δ_{13} is the Dirac phase ($0 \leq \delta_{13} < 2\pi$). The diagonal unitary matrix D^M can be written as

$$D^M = \text{diag}(1, e^{i\lambda_{21}}, e^{i\lambda_{31}}), \quad (2.29)$$

in terms of the two Majorana phases λ_{21} and λ_{31} ,

All the phases in the mixing matrix violate the CP symmetry (see Giunti and Kim (2007); Branco *et al.* (2012)). Since the Majorana phases are observable only in processes which are allowed only in the case of Majorana neutrinos, as neutrinoless double- β decay, in most

observable processes CP violation is generated by the Dirac phase. The size of this CP violation can be quantified in a parameterization-invariant way by the Jarlskog invariant (Jarlskog, 1985b,a; Greenberg, 1985; Dunietz *et al.*, 1985; Wu, 1986; Krastev and Petcov, 1988)

$$J \equiv \text{Im}(U_{e2}U_{e3}^*U_{\mu 2}^*U_{\mu 3}) = \text{Im}(U_{e2}^DU_{e3}^DU_{\mu 2}^DU_{\mu 3}^D). \quad (2.30)$$

Using the unitarity of the mixing matrix, it can be shown that other combinations of flavor and mass indices give the same result, up to a sign:

$$\text{Im}(U_{\alpha k}U_{\alpha j}^*U_{\beta k}^*U_{\beta j}) = s_{\alpha\beta kj} J, \quad (2.31)$$

with $s_{\alpha\beta kj} = \pm 1$. In the standard parameterization in Eq. (2.28), we have

$$J = c_{12}s_{12}c_{23}s_{23}c_{13}^2s_{13}\sin\delta_{13}. \quad (2.32)$$

Let us also note that in the leptonic weak neutral current,

$$j_{\text{NC}}^\rho = \sum_{\alpha=e,\mu,\tau} \overline{\nu_{\alpha L}} \gamma^\rho \nu_{\alpha L} = \sum_{k,j=1}^3 \overline{\nu_{kL}} \gamma^\rho \nu_{jL}, \quad (2.33)$$

the unitarity of U implies the absence of neutral-current transitions among different massive neutrinos (GIM mechanism; Glashow *et al.* (1970)).

D. Neutrino oscillations

Flavor neutrinos are produced and detected in charged-current weak interaction processes described by the leptonic current in Eq. (2.26). Hence, a neutrino with flavor $\alpha = e, \mu, \tau$ created in a charged-current weak interaction process from a charged lepton α^- or together with a charged antilepton α^+ is described by the state

$$|\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle. \quad (2.34)$$

Since the mixing matrix is unitary, we have the inverted relation

$$|\nu_k\rangle = \sum_\alpha U_{\alpha k} |\nu_\alpha\rangle. \quad (2.35)$$

The massive neutrino states $|\nu_k\rangle$ are eigenstates of the free Hamiltonian with energy eigenvalues

$$E_k = \sqrt{|\vec{p}_k|^2 + m_k^2}, \quad (2.36)$$

where \vec{p}_k are the respective momenta. In the plane-wave approximation (see Giunti and Kim (2007)), the space-time evolution of a massive neutrino is given by

$$|\nu_k(\vec{L}, T)\rangle = e^{-iE_k T + i\vec{p}_k \cdot \vec{L}} |\nu_k\rangle, \quad (2.37)$$

where (\vec{L}, T) is the space-time distance from the production point. Inserting this equation in Eq. (2.34) and using Eq. (2.35), we obtain

$$\begin{aligned} |\nu_\alpha(\vec{L}, T)\rangle &= \sum_k U_{\alpha k}^* e^{-iE_k T + i\vec{p}_k \cdot \vec{L}} |\nu_k\rangle \\ &= \sum_{\beta=e,\mu,\tau} \left(\sum_k U_{\alpha k}^* e^{-iE_k T + i\vec{p}_k \cdot \vec{L}} U_{\beta k} \right) |\nu_\beta\rangle. \end{aligned} \quad (2.38)$$

Then, the phase differences of different massive neutrinos generate flavor transitions with probability

$$\begin{aligned} P_{\nu_\alpha \rightarrow \nu_\beta}(\vec{L}, T) &= |\langle \nu_\beta | \nu_\alpha(\vec{L}, T) \rangle|^2 \\ &= \left| \sum_k U_{\alpha k}^* e^{-iE_k T + i\vec{p}_k \cdot \vec{L}} U_{\beta k} \right|^2. \end{aligned} \quad (2.39)$$

Since the source-detector distance $L \equiv |\vec{L}|$ is macroscopic, we can consider all massive neutrino momenta \vec{p}_k aligned along \vec{L} . Moreover, taking into account the smallness of neutrino masses, in oscillation experiments in which the neutrino propagation time T is not measured it is possible to approximate $T = L$ (see Giunti and Kim (2007)). With these approximations, the phases in Eq. (2.39) reduce to

$$\begin{aligned} -E_k T + p_k L &= -(E_k - p_k) L = -\frac{E_k^2 - p_k^2}{E_k + p_k} L \\ &= -\frac{m_k^2}{E_k + p_k} L \simeq -\frac{m_k^2}{2E_\nu} L, \end{aligned} \quad (2.40)$$

at lowest order in the neutrino masses. Here, $p_k \equiv |\vec{p}_k|$ and E_ν is the neutrino energy neglecting mass contributions. Equation (2.40) shows that the phases of massive neutrinos relevant for oscillations are independent of the values of the energies and momenta of different massive neutrinos, because of the relativistic dispersion relation in Eq. (2.36). The flavor transition probabilities are

$$\begin{aligned} P_{\nu_\alpha \rightarrow \nu_\beta}(L, E_\nu) &= \delta_{\alpha\beta} \\ &- 4 \sum_{k>j} \text{Re}(U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*) \sin^2 \left(\frac{\Delta m_{kj}^2 L}{4E_\nu} \right) \\ &- 2 \sum_{k>j} \text{Im}(U_{\alpha k} U_{\alpha j}^* U_{\beta k}^* U_{\beta j}) \sin \left(\frac{\Delta m_{kj}^2 L}{2E_\nu} \right). \end{aligned} \quad (2.41)$$

where $\Delta m_{kj}^2 = m_k^2 - m_j^2$.

Since the mixing of antineutrinos is given by (see Giunti and Kim (2007))

$$|\bar{\nu}_\alpha\rangle = \sum_k U_{\alpha k} |\bar{\nu}_k\rangle, \quad (2.42)$$

The CP-conjugated oscillation probabilities are given by

$$P_{\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta} = P_{\nu_\alpha \rightarrow \nu_\beta} |_{U \rightarrow U^*}. \quad (2.43)$$

The CPT symmetry implies that the T-conjugated and CP-conjugated oscillation probabilities are equal:

$$P_{\nu_\beta \rightarrow \nu_\alpha} = P_{\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta}. \quad (2.44)$$

In particular, the survival probabilities ($\alpha = \beta$) are CP-invariant:

$$P_{\bar{\nu}_\alpha \rightarrow \bar{\nu}_\alpha} = P_{\nu_\alpha \rightarrow \nu_\alpha}. \quad (2.45)$$

CP and T violations are observable in flavor transitions ($\alpha \neq \beta$) by measuring the asymmetries

$$\begin{aligned} A_{\alpha\beta}^{\text{CP}} &\equiv P_{\nu_\alpha \rightarrow \nu_\beta} - P_{\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta} = P_{\nu_\alpha \rightarrow \nu_\beta} - P_{\nu_\beta \rightarrow \nu_\alpha} \equiv A_{\alpha\beta}^{\text{T}} \\ &= -4J \sum_{k>j} s_{\alpha\beta kj} \sin\left(\frac{\Delta m_{kj}^2 L}{2E_\nu}\right), \end{aligned} \quad (2.46)$$

where J is the Jarlskog invariant in Eq. (2.30) and $s_{\alpha\beta kj} = \pm 1$ is given by Eq. (2.31).

In the approximation of two-neutrino mixing, in which one of the three massive neutrino components of two flavor neutrinos is neglected, the mixing matrix reduces to

$$U = \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix}, \quad (2.47)$$

where ϑ is the mixing angle ($0 \leq \vartheta \leq \pi/2$). In this approximation, there is only one squared-mass difference Δm^2 and the transition probability is given by

$$P_{\nu_\alpha \rightarrow \nu_\beta}^{2\nu}(L, E_\nu) = \sin^2 2\vartheta \sin^2\left(\frac{\Delta m^2 L}{4E_\nu}\right) \quad (\alpha \neq \beta). \quad (2.48)$$

The corresponding survival probabilities are given by

$$P_{\nu_\alpha \rightarrow \nu_\alpha}^{2\nu}(L, E_\nu) = 1 - \sin^2 2\vartheta \sin^2\left(\frac{\Delta m^2 L}{4E_\nu}\right). \quad (2.49)$$

These simple expressions are often used in the analysis of experimental data.

When neutrinos propagate in matter, the potential generated by the coherent forward elastic scattering with the particles in the medium (electrons and nucleons) modifies mixing and oscillations (Wolfenstein, 1978). In a medium with varying density it is possible to have resonant flavor transitions (Mikheev and Smirnov, 1985). This is the famous MSW effect.

The effective potentials for ν_α and $\bar{\nu}_\alpha$ are, respectively,

$$V_\alpha = V_{\text{CC}} \delta_{\alpha e} + V_{\text{NC}}, \quad \bar{V}_\alpha = -V_\alpha, \quad (2.50)$$

with the charged-current and neutral-current potentials

$$V_{\text{CC}} = \sqrt{2} G_F N_e, \quad V_{\text{NC}} = -\frac{1}{2} \sqrt{2} G_F N_n, \quad (2.51)$$

generated, respectively, by the Feynman diagrams in Fig. 1(a) and 2(a). Here N_e and N_n are the electron

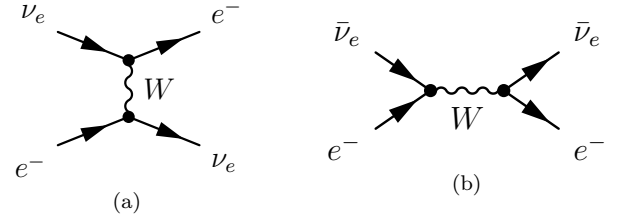


FIG. 1 Feynman diagrams of the coherent charged-current forward elastic scattering processes that generate the potentials V_{CC} (a) and \bar{V}_{CC} (b).

and neutron number densities in the medium (in an electrically neutral medium the neutral-current potentials of protons and electrons cancel each other). In normal matter, these potentials are very small, because

$$\sqrt{2} G_F \simeq 7.63 \times 10^{-14} \frac{\text{eV cm}^3}{N_A}, \quad (2.52)$$

where N_A is Avogadro's number given in Eq. (A1).

Let us consider, for simplicity, two-neutrino ν_e - ν_a mixing, where ν_a is a linear combination of ν_μ and ν_τ (which can be pure ν_μ or ν_τ as special cases). This is a good approximation for solar neutrinos. In general, a neutrino produced at $x = 0$ is described at a distance x by a state

$$|\nu(x)\rangle = \varphi_e(x) |\nu_e\rangle + \varphi_a(x) |\nu_a\rangle. \quad (2.53)$$

Taking into account that for ultrarelativistic neutrinos the distance x is approximately equal to the propagation time t , the evolution of the flavor amplitudes $\varphi_e(x)$ and $\varphi_a(x)$ with the distance x is given by the Schrödinger equation (Wolfenstein, 1978)

$$i \frac{d}{dx} \begin{pmatrix} \varphi_e(x) \\ \varphi_a(x) \end{pmatrix} = H \begin{pmatrix} \varphi_e(x) \\ \varphi_a(x) \end{pmatrix}, \quad (2.54)$$

with the effective Hamiltonian matrix

$$H = \frac{1}{4E_\nu} \begin{pmatrix} -\Delta m^2 \cos 2\vartheta + A_{\text{CC}} & \Delta m^2 \sin 2\vartheta \\ \Delta m^2 \sin 2\vartheta & \Delta m^2 \cos 2\vartheta - A_{\text{CC}} \end{pmatrix}, \quad (2.55)$$

where $A_{\text{CC}} = 2E_\nu V_{\text{CC}}$. In Eq. (2.55) we took into account only the difference V_{CC} of the potentials of ν_e and ν_a , which affects neutrino oscillations. In the framework of three-neutrino mixing the neutral-current potential V_{NC} , which is common to the three neutrino flavors, does not have any effect. However, one must be aware that the neutral-current potential V_{NC} must be taken into account in extensions of three-neutrino mixing involving sterile states and/or spin-flavor transitions (see Section VI.B).

For an initial ν_e , as in the case of solar neutrinos, the boundary condition for the solution of the differential equation is

$$\begin{pmatrix} \varphi_e(0) \\ \varphi_a(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (2.56)$$

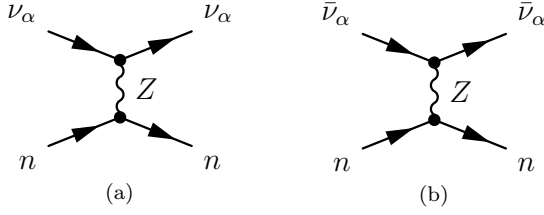


FIG. 2 Feynman diagram of the coherent neutral-current forward elastic scattering processes that generate the potentials V_{NC} (a) and \bar{V}_{NC} (b).

and the probabilities of $\nu_e \rightarrow \nu_a$ transitions and ν_e survival are, respectively,

$$P_{\nu_e \rightarrow \nu_a}(x) = |\varphi_a(x)|^2, \quad (2.57)$$

$$P_{\nu_e \rightarrow \nu_e}(x) = |\varphi_e(x)|^2 = 1 - P_{\nu_e \rightarrow \nu_a}(x). \quad (2.58)$$

The effective Hamiltonian matrix in Eq. (2.55) can be diagonalized with the transformation

$$\begin{pmatrix} \varphi_e(x) \\ \varphi_a(x) \end{pmatrix} = U_M \begin{pmatrix} \varphi_1^M(x) \\ \varphi_2^M(x) \end{pmatrix}, \quad (2.59)$$

with the effective orthogonal ($U_M^T = U_M^{-1}$) mixing matrix in matter

$$U_M = \begin{pmatrix} \cos \vartheta_M & \sin \vartheta_M \\ -\sin \vartheta_M & \cos \vartheta_M \end{pmatrix}, \quad (2.60)$$

such that

$$U_M^T H U_M = \frac{\text{diag}(-\Delta m_M^2, \Delta m_M^2)}{4E_\nu}. \quad (2.61)$$

The amplitudes $\varphi_1^M(x)$ and $\varphi_2^M(x)$ correspond to the effective massive neutrinos in matter $\nu_1^M(x)$ and $\nu_2^M(x)$, which have the effective squared-mass difference

$$\Delta m_M^2 = \sqrt{(\Delta m^2 \cos 2\vartheta - 2E_\nu V_{\text{CC}})^2 + (\Delta m^2 \sin 2\vartheta)^2}. \quad (2.62)$$

The effective mixing angle in matter ϑ_M is given by

$$\tan 2\vartheta_M = \frac{\tan 2\vartheta}{1 - \frac{2E_\nu V_{\text{CC}}}{\Delta m^2 \cos 2\vartheta}}. \quad (2.63)$$

The most interesting characteristic of this expression is that there is a resonance (Mikheev and Smirnov, 1985) when

$$V_{\text{CC}} = \frac{\Delta m^2}{2E_\nu} \cos 2\vartheta, \quad (2.64)$$

which corresponds to the electron number density

$$N_e^{\text{R}} = \frac{\Delta m^2 \cos 2\vartheta}{2\sqrt{2}E_\nu G_{\text{F}}}. \quad (2.65)$$

At the resonance the effective mixing angle is equal to $\pi/4$, i.e. the mixing is maximal, leading to the possibility of total transitions between the two flavors if the resonance region is wide enough.

In general, the evolution equation (2.54) must be solved numerically or with appropriate approximations. In a constant matter density, it is easy to derive an analytic solution, leading to the transition probability

$$P_{\nu_e \rightarrow \nu_a}^{2\nu}(x) = \sin^2 2\vartheta_M \sin^2 \left(\frac{\Delta m_M^2 x}{4E_\nu} \right). \quad (2.66)$$

This expression has the same structure as the two-neutrino transition probability in vacuum in Eq. (2.48), with the mixing angle and the squared-mass difference replaced by their effective values in matter.

The matter effect is especially important for solar neutrinos, which are created as electron neutrinos by thermonuclear reactions in the center of the Sun, where the electron number density N_e is of the order of $10^2 N_{\text{A}} \text{ cm}^{-3}$, and propagate out of the Sun through an electron density which decreases approximately in an exponential way (see Giunti and Kim (2007)). In a first approximation which neglects the small effects due to ϑ_{13} , ν_e is mixed only with ν_1 and ν_2 , which are almost equally mixed with ν_μ and ν_τ (see Section II.E). In this approximation, the oscillations of solar neutrinos are well described by the two-neutrino ν_e - ν_a mixing formalism, with oscillations generated by the solar squared-mass difference

$$\Delta m_S^2 \approx 8 \times 10^{-5} \text{ eV}^2, \quad (2.67)$$

and

$$\begin{aligned} |\nu_a\rangle &\simeq \cos \vartheta_{23} |\nu_\mu\rangle - \sin \vartheta_{23} |\nu_\tau\rangle \\ &\approx (|\nu_\mu\rangle - |\nu_\tau\rangle)/\sqrt{2}. \end{aligned} \quad (2.68)$$

An electron neutrino created in the center of the Sun is the linear combination of effective massive neutrinos

$$|\nu_e^0\rangle = \cos \vartheta_M^0 |\nu_1^0\rangle + \sin \vartheta_M^0 |\nu_2^0\rangle, \quad (2.69)$$

where ν_1^0 and ν_2^0 are the effective massive neutrinos at the point of neutrino production near the center of the Sun and ϑ_M^0 is the corresponding effective mixing angle. Since the resonance is crossed adiabatically, there are no transitions between the effective massive neutrinos during propagation and the state which emerges from the Sun is

$$|\nu_S\rangle = \cos \vartheta_M^0 |\nu_1\rangle + \sin \vartheta_M^0 |\nu_2\rangle, \quad (2.70)$$

where ν_1 and ν_2 are the massive neutrinos in vacuum. Since the two massive neutrinos lose coherence during the long propagation from the Sun to the Earth (Dighe

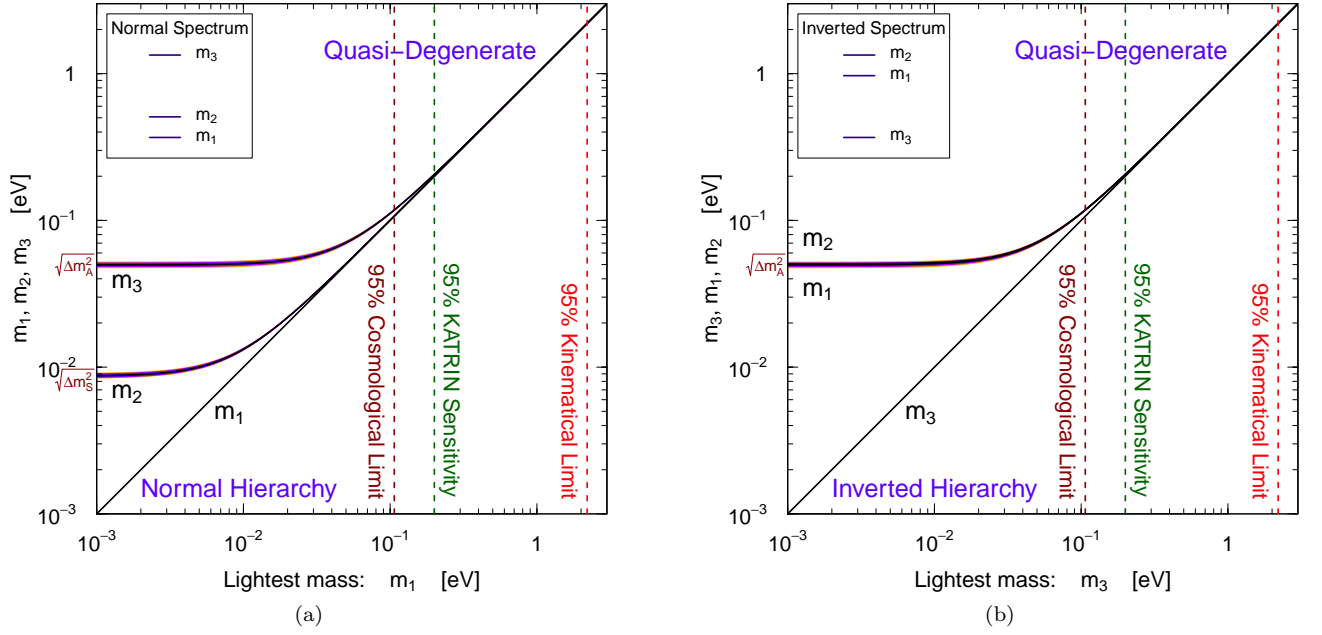


FIG. 3 (a) Values of the neutrino masses as functions of the lightest mass m_1 in the normal mass spectrum obtained with the squared-mass differences in Tab. II. (b) Corresponding values of the neutrino masses as functions of the lightest mass m_3 in the inverted mass spectrum.

et al., 1999), experiments on Earth measure the average electron neutrino survival probability (Parke, 1986)

$$\begin{aligned} \bar{P}_{\nu_e \rightarrow \nu_e}^{S, 2\nu} &= \cos^2 \vartheta_M^0 |\langle \nu_e | \nu_1 \rangle|^2 + \sin^2 \vartheta_M^0 |\langle \nu_e | \nu_2 \rangle|^2 \\ &= \frac{1}{2} + \frac{1}{2} \cos 2\vartheta_M^0 \cos 2\vartheta. \end{aligned} \quad (2.71)$$

This is a surprisingly simple expression, which depends only on the mixing angle in vacuum ϑ and on the effective mixing angle in the center of the Sun ϑ_M^0 , which can be easily calculated using Eq. (2.63). Notice that ϑ_M^0 depends on the neutrino energy. With the value of Δm_S^2 in Eq. (2.67), $\vartheta_M^0 \simeq \vartheta$ for $E_\nu \lesssim 1$ MeV and $\vartheta_M^0 \simeq \pi/2$ for $E_\nu \gtrsim 5$ MeV (see Giunti and Kim (2007)). Therefore,

$$\bar{P}_{\nu_e \rightarrow \nu_e}^{S, 2\nu} \simeq \begin{cases} 1 - 0.5 \sin^2 2\vartheta & \text{for } E_\nu \lesssim 1 \text{ MeV,} \\ \sin^2 \vartheta & \text{for } E_\nu \gtrsim 5 \text{ MeV.} \end{cases} \quad (2.72)$$

E. Status of three-neutrino mixing

The results of several solar, atmospheric and long-baseline neutrino oscillation experiments have proved that neutrinos are massive and mixed particles (see Giunti and Kim (2007); Bilenky (2010); Xing and Zhou (2011); Gonzalez-Garcia *et al.* (2012); Bellini *et al.* (1960); NuFIT (2013); Capozzi *et al.* (2013)). There are two groups of experiments which measured two types of flavor transition generated by two independent squared-mass differences (Δm^2): the solar squared-mass differ-

ence in Eq. (2.67) and the atmospheric squared-mass difference

$$\Delta m_A^2 \approx 2 \times 10^{-3} \text{ eV}^2. \quad (2.73)$$

Since in the framework of three-neutrino mixing described in Section II.C there are just two independent squared-mass differences, solar, atmospheric and long-baseline data have led us to the current three-neutrino mixing paradigm, with the standard assignments

$$\Delta m_S^2 = \Delta m_{21}^2 \ll \Delta m_A^2 = \frac{1}{2} |\Delta m_{31}^2 + \Delta m_{32}^2|. \quad (2.74)$$

The absolute value in the definition of Δm_A^2 is necessary, because there are the two possible spectra for the neutrino masses illustrated schematically in the insets of the two corresponding panels in Fig. 3: the normal spectrum (NS) with $m_1 < m_2 < m_3$ and $\Delta m_{13}^2, \Delta m_{23}^2 > 0$; the inverted spectrum (IS) with $m_3 < m_1 < m_2$ and $\Delta m_{13}^2, \Delta m_{23}^2 < 0$.

The three-neutrino mixing parameters can be determined with good precision with a global fit of neutrino oscillation data. In Tab. II we report the results of the latest global fit presented in Capozzi *et al.* (2013), which agree, within the uncertainties, with the NuFIT-v1.2 (NuFIT, 2013) update of the global analysis presented in Gonzalez-Garcia *et al.* (2012). One can see that all the oscillation parameters are determined with precision between about 3% and 11%. The largest uncertainty is that of ϑ_{23} , which is known to be close to maximal ($\pi/4$),

but it is not known if it is smaller or larger than $\pi/4$. For the Dirac CP-violating phase δ , there is an indication in favor of $\delta \approx 3\pi/2$, which would give maximal CP violation, but at 3σ all the values of δ are allowed, including the CP-conserving values $\delta = 0, \pi$.

An open problem in the framework of three-neutrino mixing is the determination of the absolute scale of neutrino masses, which is cannot be determined with neutrino oscillation experiments, because oscillations depend only on the differences of neutrino masses. However, the measurement in neutrino oscillation experiments of the neutrino squared-mass differences allows us to constraint the allowed patterns of neutrino masses. A convenient way to see the allowed patterns of neutrino masses is to plot the values of the masses as functions of the unknown lightest mass, which is m_1 in the normal mass spectrum and m_3 in the inverted spectrum, as shown in Figs. 3. We used the squared-mass differences in Tab. II. Figure 3 shows that there are three extreme possibilities:

A normal hierarchy: $m_1 \ll m_2 \ll m_3$. In this case

$$m_2 \simeq \sqrt{\Delta m_S^2} \approx 9 \times 10^{-3} \text{ eV}, \quad (2.75)$$

$$m_3 \simeq \sqrt{\Delta m_A^2} \approx 5 \times 10^{-2} \text{ eV}. \quad (2.76)$$

An inverted hierarchy: $m_3 \ll m_1 \lesssim m_2$ In this case

$$m_1 \lesssim m_2 \simeq \sqrt{\Delta m_A^2} \approx 5 \times 10^{-2} \text{ eV}. \quad (2.77)$$

Quasi-degenerate spectra: $m_1 \lesssim m_2 \lesssim m_3 \simeq m_\nu$ in the normal scheme and $m_3 \lesssim m_1 \lesssim m_2 \simeq m_\nu$ in the inverted scheme, with

$$m_\nu \gg \sqrt{\Delta m_A^2} \approx 5 \times 10^{-2} \text{ eV}. \quad (2.78)$$

There are three main sources of information on the absolute scale of neutrino masses:

Beta decay: The most robust information on neutrino masses can be obtained in β -decay experiments which measure the kinematical effect of neutrino masses on the energy spectrum of the emitted electron. Tritium β -decay experiments obtained the most stringent bounds on the neutrino masses by limiting the effective electron neutrino mass m_β given by (see Giunti and Kim (2007); Bilenky (2010); Xing and Zhou (2011))

$$m_\beta^2 = \sum_{k=1}^3 |U_{ek}|^2 m_k^2. \quad (2.79)$$

The most stringent 95% CL limits obtained in the Mainz (Kraus *et al.*, 2005) and Troitsk (Aseev *et al.*, 2011) experiments,

$$m_\beta \leq 2.3 \text{ eV} \quad (\text{Mainz}), \quad (2.80)$$

$$m_\beta \leq 2.1 \text{ eV} \quad (\text{Troitsk}), \quad (2.81)$$

are shown in Fig. 3. The KATRIN experiment (Fraenkle, 2011), which is scheduled to start data taking in 2014, is expected to have a sensitivity to m_β of about 0.2 eV (also shown in Fig. 3).

Neutrinoless double-beta decay: This process occurs only if massive neutrinos are Majorana fermions and depends on the effective Majorana mass (see Giunti and Kim (2007); Bilenky (2010); Xing and Zhou (2011); Bilenky and Giunti (2012))

$$m_{\beta\beta} = \left| \sum_{k=1}^3 U_{ek}^2 m_k \right|. \quad (2.82)$$

The most stringent limits, have been obtained combining the results of EXO (Auger *et al.*, 2012) and KamLAND-Zen (Gando *et al.*, 2013) experiments with ^{136}Xe ,

$$m_{\beta\beta} \lesssim 0.12 - 0.25 \text{ eV} \quad (90\% \text{CL}), \quad (2.83)$$

and combining the results of Heidelberg-Moscow (Klapdor-Kleingrothaus *et al.*, 2001), IGEX (Aalseth *et al.*, 2002) and GERDA (Agostini *et al.*, 2013) with ^{76}Ge ,

$$m_{\beta\beta} \lesssim 0.2 - 0.4 \text{ eV} \quad (90\% \text{CL}). \quad (2.84)$$

The intervals are caused by nuclear physics uncertainties (see Vergados *et al.* (2012)).

Cosmology: Since light massive neutrinos are hot dark matter, cosmological data give information on the sum of neutrino masses (see Giunti and Kim (2007); Bilenky (2010); Xing and Zhou (2011); Lesgourgues *et al.* (2013)). The analysis of cosmological data in the framework of the standard Cold Dark Matter model with a cosmological constant (Λ CDM) disfavors neutrino masses larger than some fraction of eV, but the value of the upper bound on the sum of neutrino masses depends on model assumptions and on the considered data set (see Wong (2011)). Figure 3 shows the 95% limit

$$\sum_{k=1}^3 m_k < 0.32 \text{ eV}, \quad (2.85)$$

obtained recently by the Planck collaboration (Ade *et al.*, 2013).

III. ELECTROMAGNETIC FORM FACTORS

The importance of neutrino electromagnetic properties was first mentioned by Pauli in 1930, when he postulated the existence of this particle and discussed the possibility that the neutrino might have a magnetic moment. Systematic theoretical studies of neutrino electromagnetic properties started after it was shown that in

Parameter	Spectrum	Best fit	1 σ range	2 σ range	3 σ range	rel. unc.
$\Delta m_{\text{S}}^2/10^{-5} \text{ eV}^2$		7.54	7.32 – 7.80	7.15 – 8.00	6.99 – 8.18	3%
$\sin^2 \vartheta_{12}/10^{-1}$		3.08	2.91 – 3.25	2.75 – 3.42	2.59 – 3.59	5%
$\Delta m_{\text{A}}^2/10^{-3} \text{ eV}^2$	NS	2.44	2.38 – 2.52	2.30 – 2.59	2.22 – 2.66	3%
	IS	2.40	2.33 – 2.47	2.25 – 2.54	2.17 – 2.61	3%
$\sin^2 \vartheta_{23}/10^{-1}$	NS	4.25	3.98 – 4.54	3.76 – 5.06	3.57 – 6.41	11%
	IS	4.37	4.08 – 6.10	3.84 – 6.37	3.63 – 6.59	11%
$\sin^2 \vartheta_{13}/10^{-2}$	NS	2.34	2.16 – 2.56	1.97 – 2.76	1.77 – 2.97	9%
	IS	2.39	2.18 – 2.60	1.98 – 2.80	1.78 – 3.00	9%

TABLE II Values of the neutrino mixing parameters obtained with global analysis of neutrino oscillation data presented in Capozzi *et al.* (2013) in the framework of three-neutrino mixing with a normal spectrum (NS) or an inverted spectrum (IS). The relative uncertainty (rel. unc.) has been obtained from the 3 σ range divided by 6.

the extended Standard Model with right-handed neutrinos the magnetic moment of a massive neutrino is, in general, nonvanishing and that its value is determined by the neutrino mass (Marciano and Sanda, 1977; Lee and Shrock, 1977; Fujikawa and Shrock, 1980; Petcov, 1977; Pal and Wolfenstein, 1982; Shrock, 1982; Bilenky and Petcov, 1987).

Neutrino electromagnetic properties are important because they are directly connected to fundamentals of particle physics. For example, neutrino electromagnetic properties can be used to distinguish Dirac and Majorana neutrinos (see Schechter and Valle (1981); Shrock (1982); Pal and Wolfenstein (1982); Nieves (1982); Kayser (1982, 1984)) and also as probes of new physics that might exist beyond the Standard Model (see Bell *et al.* (2005, 2006); Bell (2007); Novales-Sanchez *et al.* (2008)).

In this Section we discuss the general form of the electromagnetic interactions of Dirac and Majorana neutrinos in the one-photon approximation. In Subsection III.A we derive the general expression of the effective electromagnetic coupling of Dirac neutrinos in terms of electromagnetic form factors and we discuss the properties of the form factors under CP and CPT transformations. In Subsection III.B we consider Majorana neutrinos and in Subsection III.C we consider the Standard Model case of massless Weyl neutrinos. In Subsection III.D we discuss the derivation of the form factors in gauge models.

A. Dirac neutrinos

In the Standard Model, the interaction of a fermionic field $\mathbf{f}(x)$ with the electromagnetic field $A^\mu(x)$ is given by the interaction Hamiltonian

$$\mathcal{H}_{\text{em}}^{(\mathbf{f})}(x) = j_\mu^{(\mathbf{f})}(x) A^\mu(x) = q_{\mathbf{f}} \bar{\mathbf{f}}(x) \gamma_\mu \mathbf{f}(x) A^\mu(x), \quad (3.1)$$

where $q_{\mathbf{f}}$ is the charge of the fermion \mathbf{f} . Figure 4(a) shows the corresponding tree-level Feynman diagram (the photon γ is the quantum of the electromagnetic field $A^\mu(x)$).

For neutrinos the electric charge is zero and there are

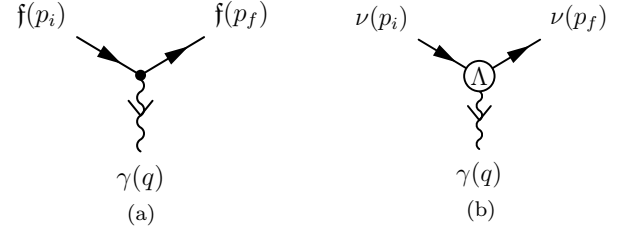


FIG. 4 Tree-level coupling of a charged fermion \mathbf{f} with a photon γ (a) and effective one-photon coupling of a neutrino with a photon (b).

no electromagnetic interactions at tree-level¹. However, such interactions can arise at the quantum level from loop diagrams at higher order of the perturbative expansion of the interaction. In the one-photon approximation², the electromagnetic interactions of a neutrino field $\nu(x)$ can be described by the effective interaction Hamiltonian

$$\mathcal{H}_{\text{em}}^{(\nu)}(x) = j_\mu^{(\nu)}(x) A^\mu(x) = \bar{\nu}(x) \Lambda_\mu \nu(x) A^\mu(x), \quad (3.2)$$

where, $j_\mu^{(\nu)}(x)$ is the effective neutrino electromagnetic current four-vector and Λ_μ is a 4×4 matrix in spinor space which can contain space-time derivatives, such that $j_\mu^{(\nu)}(x)$ transforms as a four-vector. Since radiative corrections are generated by weak interactions which are not invariant under a parity transformation, $j_\mu^{(\nu)}(x)$ can be a sum of polar and axial parts. The corresponding diagram for the interaction of a neutrino with a photon is shown in Fig. 4(b), where the blob represents the quantum loop contributions.

As we will see in the following, the neutrino electromagnetic properties corresponding to the diagram in Fig. 4(b) include charge and magnetic form factors. Let us emphasize that these neutrino electromagnetic properties can exist even if neutrinos are elementary particles,

¹ However, in some theories beyond the Standard Model neutrinos can be millicharged particles (see Section VII.A).

² Some cases in which the one-photon approximation breaks down are discussed in Section VII.A.

without an internal structure, because they are generated by quantum loop effects. Thus, the neutrino charge and magnetic form factors have a different origin from the neutron charge and magnetic form factors (also called Dirac and Pauli form factors), which are mainly due to its internal quark structure. For example, the neutrino magnetic moment (which is the magnetic form factor for interactions with real photons, i.e. $q^2 = 0$ in Fig. 4(b)) have the same quantum origin as the anomalous magnetic moment of the electron (see Greiner and Reinhardt (2009)).

We are interested in the neutrino part of the amplitude corresponding to the diagram in Fig. 4(b), which is given by the matrix element

$$\langle \nu(p_f, h_f) | j_\mu^{(\nu)}(x) | \nu(p_i, h_i) \rangle, \quad (3.3)$$

where p_i (p_f) and h_i (h_f) are the four-momentum and helicity of the initial (final) neutrino. Taking into account that

$$\partial^\mu j_\mu^{(\nu)}(x) = i [\mathcal{P}^\mu, j_\mu^{(\nu)}(x)], \quad (3.4)$$

where \mathcal{P}^μ is the four-momentum operator which generate translations, the effective current can be written as

$$j_\mu^{(\nu)}(x) = e^{i\mathcal{P} \cdot x} j_\mu^{(\nu)}(0) e^{-i\mathcal{P} \cdot x}. \quad (3.5)$$

Since $\mathcal{P}^\mu |\nu(p)\rangle = p^\mu |\nu(p)\rangle$, we have

$$\langle \nu(p_f) | j_\mu^{(\nu)}(x) | \nu(p_i) \rangle = e^{i(p_f - p_i) \cdot x} \langle \nu(p_f) | j_\mu^{(\nu)}(0) | \nu(p_i) \rangle, \quad (3.6)$$

where we suppressed for simplicity the helicity labels which are not of immediate relevance. Here we see that the unknown quantity which determines the neutrino-photon interaction is $\langle \nu(p_f) | j_\mu^{(\nu)}(0) | \nu(p_i) \rangle$. Considering that the incoming and outgoing neutrinos are free particles which are described by free Dirac fields with the Fourier expansion in Eq. (A53), we have

$$\langle \nu(p_f) | j_\mu^{(\nu)}(0) | \nu(p_i) \rangle = \bar{u}(p_f) \Lambda_\mu(p_f, p_i) u(p_i). \quad (3.7)$$

The electromagnetic properties of neutrinos are embodied by $\Lambda_\mu(p_f, p_i)$, which is a matrix in spinor space and can be decomposed in terms of linearly independent products of Dirac γ matrices and the available kinematical four-vectors p_i and p_f . As shown in Appendix B, the most general decomposition can be written as

$$\Lambda_\mu(p_f, p_i) = \mathbb{f}_1(q^2) q_\mu + \mathbb{f}_2(q^2) q_\mu \gamma_5 + \mathbb{f}_3(q^2) \gamma_\mu + \mathbb{f}_4(q^2) \gamma_\mu \gamma_5 + \mathbb{f}_5(q^2) \sigma_{\mu\nu} q^\nu + \mathbb{f}_6(q^2) \epsilon_{\mu\nu\rho\gamma} q^\nu \sigma^{\rho\gamma}, \quad (3.8)$$

where $\mathbb{f}_k(q^2)$ are six Lorentz-invariant form factors ($k = 1, \dots, 6$) and q is the four-momentum of the photon, which is given by

$$q = p_i - p_f, \quad (3.9)$$

from energy-momentum conservation. Notice that the form factors depend only on q^2 , which is the only available Lorentz-invariant kinematical quantity, since $(p_i + p_f)^2 = 4m^2 - q^2$. Therefore, $\Lambda_\mu(p_f, p_i)$ depends only on q and from now on we will denote it as $\Lambda_\mu(q)$.

Since the Hamiltonian and the electromagnetic field are Hermitian ($\mathcal{H}_{\text{em}}^{(\nu)\dagger} = \mathcal{H}_{\text{em}}^{(\nu)}$ and $A^{\mu\dagger} = A^\mu$), the effective current must be Hermitian, $j_\mu^{(\nu)\dagger} = j_\mu^{(\nu)}$. Hence, we have

$$\langle \nu(p_f) | j_\mu^{(\nu)}(0) | \nu(p_i) \rangle = \langle \nu(p_i) | j_\mu^{(\nu)}(0) | \nu(p_f) \rangle^*, \quad (3.10)$$

which leads to

$$\Lambda_\mu(q) = \gamma^0 \Lambda_\mu^\dagger(-q) \gamma^0. \quad (3.11)$$

Using the properties of the Dirac matrices (see Appendix A), one can find that this constraint implies that

$$\mathbb{f}_2, \quad \mathbb{f}_3, \quad \mathbb{f}_4 \quad \text{are real}, \quad (3.12)$$

and

$$\mathbb{f}_1, \quad \mathbb{f}_5, \quad \mathbb{f}_6 \quad \text{are imaginary}. \quad (3.13)$$

The number of independent form factors can be reduced by imposing current conservation, $\partial^\mu j_\mu^{(\nu)}(x) = 0$, which is required by gauge invariance (i.e. invariance of $\mathcal{H}_{\text{em}}^{(\nu)}(x)$ under the transformation $A^\mu(x) \rightarrow A^\mu(x) + \partial^\mu \varphi(x)$ for any $\varphi(x)$, which leaves invariant the electromagnetic tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$). Using Eq. (3.4), current conservation implies that

$$\langle \nu(p_f) | [\mathcal{P}^\mu, j_\mu^{(\nu)}(0)] | \nu(p_i) \rangle = 0. \quad (3.14)$$

Hence, in momentum space we have the constraint

$$q^\mu \bar{u}(p_f) \Lambda_\mu(q) u(p_i) = 0, \quad (3.15)$$

which implies that

$$\mathbb{f}_1(q^2) q^2 + \mathbb{f}_2(q^2) q^2 \gamma_5 + 2m \mathbb{f}_4(q^2) \gamma_5 = 0. \quad (3.16)$$

Since γ_5 and the unity matrix are linearly independent, we obtain the constraints

$$\mathbb{f}_1(q^2) = 0, \quad \mathbb{f}_4(q^2) = -\mathbb{f}_2(q^2) q^2 / 2m. \quad (3.17)$$

Therefore, in the most general case consistent with Lorentz and electromagnetic gauge invariance, the vertex function $\Lambda_\mu(q)$ is defined in terms of four form factors (Nieves, 1982; Kayser, 1982, 1984),

$$\Lambda_\mu(q) = \mathbb{f}_Q(q^2) \gamma_\mu - \mathbb{f}_M(q^2) i \sigma_{\mu\nu} q^\nu + \mathbb{f}_E(q^2) \sigma_{\mu\nu} q^\nu \gamma_5 + \mathbb{f}_A(q^2) (q^2 \gamma_\mu - q_\mu \not{q}) \gamma_5. \quad (3.18)$$

where $\mathbb{f}_Q = \mathbb{f}_3$, $\mathbb{f}_M = i\mathbb{f}_5$, $\mathbb{f}_E = -2i\mathbb{f}_6$ and $\mathbb{f}_A = -\mathbb{f}_2/2m$ are the real charge, dipole magnetic and electric, and anapole neutrino form factors. The term involving the

electric form factor corresponds to the last term in Eq. (3.8), in which we took into account the identity in Eq. (A24). In the term involving the anapole form factor we used the identity $\bar{u}(p_f) \not{q} \gamma^5 u(p_i) = 2m \bar{u}(p_f) \gamma^5 u(p_i)$, which is easily obtained from Eqs. (A17) and (A40).

The physical meaning of the dipole magnetic and electric neutrino form factors is discussed in Section IV and that of the charge and anapole in Section VII. Here we only remark that for the coupling with a real photon ($q^2 = 0$)

$$\mathbb{f}_Q(0) = \mathbb{q}, \quad \mathbb{f}_M(0) = \mathbb{p}, \quad \mathbb{f}_E(0) = \mathbb{e}, \quad \mathbb{f}_A(0) = \mathbb{a}, \quad (3.19)$$

where \mathbb{q} , \mathbb{p} , \mathbb{e} and \mathbb{a} are, respectively, the neutrino charge, magnetic moment, electric moment and anapole moment. Although above we stated that $\mathbb{q} = 0$, here we did not enforce this equality because in some theories beyond the Standard Model neutrinos can be millicharged particles, as explained in Section VII.A.

Now it is interesting to study the properties of $\mathcal{H}_{\text{em}}^{(\nu)}(x)$ under a CP transformation, in order to find which of the terms in Eq. (3.18) violate CP. The reason is that, whereas it is well known that weak interactions violate maximally C and P, the violation of CP is a more exotic phenomenon, which has been observed so far only in the hadron sector (see Bilenky (2008)).

Using the transformation (A64) of a fermion field under an active CP transformation one can find that for the Standard Model electric current $j_\mu(x)$ in Eq. (3.1) we have

$$j_\mu(x) \xrightarrow{\text{CP}} \mathcal{U}_{\text{CP}} j_\mu(x) \mathcal{U}_{\text{CP}}^\dagger = -j^\mu(x_P). \quad (3.20)$$

Hence, the Standard Model electromagnetic interaction Hamiltonian $\mathcal{H}_{\text{em}}^{(\nu)}(x)$ is left invariant by³

$$A_\mu(x) \xrightarrow{\text{CP}} -A^\mu(x_P). \quad (3.21)$$

CP is conserved in neutrino electromagnetic interactions (in the one-photon approximation) if $j_\mu^{(\nu)}(x)$ transforms as $j_\mu(x)$:

$$\text{CP} \iff \mathcal{U}_{\text{CP}} j_\mu^{(\nu)}(x) \mathcal{U}_{\text{CP}}^\dagger = -j_\mu^{(\nu)}(x_P). \quad (3.22)$$

For the matrix element (3.7) we obtain

$$\text{CP} \iff \Lambda_\mu(q) \xrightarrow{\text{CP}} -\Lambda^\mu(q). \quad (3.23)$$

Using the formulae in Appendix A, one can find that under a CP transformation we have⁴

$$\Lambda_\mu(q) \xrightarrow{\text{CP}} \gamma^0 \mathcal{C} \Lambda_\mu^T(q_P) \mathcal{C}^\dagger \gamma^0, \quad (3.24)$$

³ The transformation $x \rightarrow x_P$ is irrelevant since all amplitudes are obtained by integrating over d^4x , as in Eq. (5.2).

⁴ The operators in $j_\mu^{(\nu)}(x)$ are implicitly assumed to be normally ordered (see Giunti and Kim (2007)).

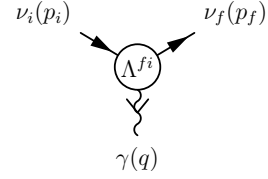


FIG. 5 Effective one-photon coupling of neutrinos with the electromagnetic field, taking into account possible transitions between two different initial and final massive neutrinos ν_i and ν_f .

with $q_P^\mu = q_\mu$. Using the form-factor expansion in Eq. (3.18), we obtain

$$\Lambda_\mu(q) \xrightarrow{\text{CP}} -[\mathbb{f}_Q(q^2) \gamma^\mu - \mathbb{f}_M(q^2) i \sigma^{\mu\nu} q_\nu - \mathbb{f}_E(q^2) \sigma^{\mu\nu} q_\nu \gamma_5 + \mathbb{f}_A(q^2) (q^2 \gamma^\mu - q^\mu \not{q}) \gamma_5]. \quad (3.25)$$

Therefore, only the electric dipole form factor violates CP:

$$\text{CP} \iff \mathbb{f}_E(q^2) = 0. \quad (3.26)$$

So far, in this Section we have considered only one massive neutrino field $\nu(x)$, but from the discussion of neutrino mixing in Section II we know that there are at least three massive neutrino fields in nature. Therefore, we must generalize the discussion to the case of N massive neutrino fields $\nu_k(x)$ with respective masses m_k ($k = 1, \dots, N$). The effective electromagnetic interaction Hamiltonian in Eq. (3.2) is generalized to

$$\mathcal{H}_{\text{em}}^{(\nu)}(x) = j_\mu^{(\nu)}(x) A^\mu(x) = \sum_{k,j=1}^N \bar{\nu}_k(x) \Lambda_\mu^{kj} \nu_j(x) A^\mu(x), \quad (3.27)$$

where we take into account possible transitions between different massive neutrinos. The physical effect of $\mathcal{H}_{\text{em}}^{(\nu)}$ is described by the effective electromagnetic vertex in Fig. 5, with the neutrino matrix element

$$\langle \nu_f(p_f) | j_\mu^{(\nu)}(0) | \nu_i(p_i) \rangle = \bar{u}_f(p_f) \Lambda_\mu^{fi}(p_f, p_i) u_i(p_i). \quad (3.28)$$

As in the case of one massive neutrino field (see Appendix B), $\Lambda_\mu^{fi}(p_f, p_i)$ depends only on the four-momentum q transferred to the photon and can be expressed in terms of six Lorentz-invariant form factors:

$$\begin{aligned} \Lambda_\mu^{fi}(q) = & \mathbb{f}_1^{fi}(q^2) q_\mu + \mathbb{f}_2^{fi}(q^2) q_\mu \gamma_5 + \mathbb{f}_3^{fi}(q^2) \gamma_\mu \\ & + \mathbb{f}_4^{fi}(q^2) \gamma_\mu \gamma_5 + \mathbb{f}_5^{fi}(q^2) \sigma_{\mu\nu} q^\nu + \mathbb{f}_6^{fi}(q^2) \epsilon_{\mu\nu\rho\gamma} q^\nu \sigma^{\rho\gamma}. \end{aligned} \quad (3.29)$$

The Hermitian nature of $j_\mu^{(\nu)}$ implies that $\langle \nu_f(p_f) | j_\mu^{(\nu)}(0) | \nu_i(p_i) \rangle = \langle \nu_i(p_i) | j_\mu^{(\nu)}(0) | \nu_f(p_f) \rangle^*$, leading to the constraint

$$\Lambda_\mu^{fi}(q) = \gamma^0 [\Lambda_\mu^{if}(-q)]^\dagger \gamma^0. \quad (3.30)$$

Considering the $N \times N$ form-factor matrices \mathbb{f}_k in the space of massive neutrinos with components \mathbb{f}_k^{fi} for $k = 1, \dots, 6$, we find that

$$\mathbb{f}_2, \mathbb{f}_3, \mathbb{f}_4 \text{ are Hermitian,} \quad (3.31)$$

and

$$\mathbb{f}_1, \mathbb{f}_5, \mathbb{f}_6 \text{ are antihermitian.} \quad (3.32)$$

Following the same method used in Eqs. (3.4)–(3.16), one can find that current conservation implies the constraints

$$\mathbb{f}_1^{fi}(q^2)q^2 + \mathbb{f}_3^{fi}(q^2)(m_f - m_i) = 0, \quad (3.33)$$

$$\mathbb{f}_2^{fi}(q^2)q^2 + \mathbb{f}_4^{fi}(q^2)(m_f + m_i) = 0. \quad (3.34)$$

Therefore, we obtain

$$\begin{aligned} \Lambda_\mu^{fi}(q) = & (\gamma_\mu - q_\mu \not{q}/q^2) \left[\mathbb{f}_Q^{fi}(q^2) + \mathbb{f}_A^{fi}(q^2)q^2\gamma_5 \right] \\ & - i\sigma_{\mu\nu}q^\nu \left[\mathbb{f}_M^{fi}(q^2) + i\mathbb{f}_E^{fi}(q^2)\gamma_5 \right], \end{aligned} \quad (3.35)$$

where $\mathbb{f}_Q^{fi} = \mathbb{f}_3^{fi}$, $\mathbb{f}_M^{fi} = i\mathbb{f}_5^{fi}$, $\mathbb{f}_E^{fi} = -2i\mathbb{f}_6^{fi}$ and $\mathbb{f}_A^{fi} = -\mathbb{f}_2^{fi}/(m_f + m_i)$, with

$$\mathbb{f}_\Omega^{fi} = (\mathbb{f}_\Omega^{if})^* \quad (\Omega = Q, M, E, A). \quad (3.36)$$

Note that since $\overline{u_f}(p_f)\not{q}u_i(p_i) = (m_f - m_i)\overline{u_f}(p_f)u_i(p_i)$, if $f = i$ Eq. (3.35) correctly reduces to Eq. (3.18).

The form-factors with $f = i$ are called “diagonal”, whereas those with $f \neq i$ are called “off-diagonal” or “transition form-factors”. This terminology follows from the expression

$$\begin{aligned} \Lambda_\mu(q) = & (\gamma_\mu - q_\mu \not{q}/q^2) \left[\mathbb{f}_Q(q^2) + \mathbb{f}_A(q^2)q^2\gamma_5 \right] \\ & - i\sigma_{\mu\nu}q^\nu \left[\mathbb{f}_M(q^2) + i\mathbb{f}_E(q^2)\gamma_5 \right], \end{aligned} \quad (3.37)$$

in which $\Lambda_\mu(q)$ is a $N \times N$ matrix in the space of massive neutrinos expressed in terms of the four Hermitian $N \times N$ matrices of form factors

$$\mathbb{f}_\Omega = \mathbb{f}_\Omega^\dagger \quad (\Omega = Q, M, E, A). \quad (3.38)$$

For the coupling with a real photon ($q^2 = 0$) we have

$$\mathbb{f}_Q^{fi}(0) = \mathbb{q}_{fi}, \mathbb{f}_M^{fi}(0) = \mathbb{p}_{fi}, \mathbb{f}_E^{fi}(0) = \mathbb{e}_{fi}, \mathbb{f}_A^{fi}(0) = \mathbb{a}_{fi}, \quad (3.39)$$

where \mathbb{q}_{fi} , \mathbb{p}_{fi} , \mathbb{e}_{fi} and \mathbb{a}_{fi} are, respectively, the neutrino charge, magnetic moment, electric moment and anapole moment of diagonal ($f = i$) and transition ($f \neq i$) types.

Considering now CP invariance, the transformation (3.22) of $j_\mu^{(\nu)}(x)$ implies the constraint in Eq. (3.23) for the $N \times N$ matrix $\Lambda_\mu(q)$ in the space of massive neutrinos. Using the formulae in Appendix A, we obtain

$$\Lambda_\mu^{fi}(q) \xrightarrow{\text{CP}} \xi_f^{\text{CP}} \xi_i^{\text{CP}*} \gamma^0 \mathcal{C} [\Lambda_\mu^{if}(q_P)]^T \mathcal{C}^\dagger \gamma^0, \quad (3.40)$$

where ξ_k^{CP} is the CP phase of ν_k . Since the massive neutrinos take part to standard charged-current weak interactions⁵, their CP phases are equal if CP is conserved (see Giunti and Kim (2007)). Hence, we have

$$\Lambda_\mu^{fi}(q) \xrightarrow{\text{CP}} \gamma^0 \mathcal{C} [\Lambda_\mu^{if}(q_P)]^T \mathcal{C}^\dagger \gamma^0. \quad (3.41)$$

Using the form-factor expansion in Eq. (3.35), we obtain

$$\begin{aligned} \Lambda_\mu^{fi}(q) \xrightarrow{\text{CP}} - \left\{ (\gamma^\mu - q^\mu \not{q}/q^2) \left[\mathbb{f}_Q^{if}(q^2) + \mathbb{f}_A^{if}(q^2)q^2\gamma_5 \right] \right. \\ \left. - i\sigma^{\mu\nu}q_\nu \left[\mathbb{f}_M^{if}(q^2) - i\mathbb{f}_E^{if}(q^2)\gamma_5 \right] \right\}. \end{aligned} \quad (3.42)$$

Imposing the constraint in Eq. (3.23), for the form factors we obtain

$$\text{CP} \iff \begin{cases} \mathbb{f}_Q^{fi} = \mathbb{f}_Q^{if} = (\mathbb{f}_Q^{fi})^* & (\Omega = Q, M, A), \\ \mathbb{f}_E^{fi} = -\mathbb{f}_E^{if} = -(\mathbb{f}_E^{fi})^*, \end{cases} \quad (3.43)$$

where, in the last equalities, we took into account the constraints (3.36). Therefore, diagonal electric form factors violate CP, in agreement with the one-generation constraint in Eq. (3.26). For the Hermitian $N \times N$ form-factor matrices, we obtain that if CP is conserved \mathbb{f}_Q , \mathbb{f}_M and \mathbb{f}_A are real and symmetric and \mathbb{f}_E is imaginary and antisymmetric:

$$\text{CP} \iff \begin{cases} \mathbb{f}_\Omega = \mathbb{f}_\Omega^T = \mathbb{f}_\Omega^* & (\Omega = Q, M, A), \\ \mathbb{f}_E = -\mathbb{f}_E^T = -\mathbb{f}_E^*. \end{cases} \quad (3.44)$$

Let us now consider antineutrinos. Using for the massive neutrino fields the Fourier expansion in Eq. (A53), the effective antineutrino matrix element for $\bar{\nu}_i(p_i) \rightarrow \bar{\nu}_f(p_f)$ transitions is given by

$$\langle \bar{\nu}_f(p_f) | j_\mu^{(\nu)}(0) | \bar{\nu}_i(p_i) \rangle = -\overline{v_i}(p_i) \Lambda_\mu^{if}(q) v_f(p_f). \quad (3.45)$$

Using the relation (A45) we can write it as

$$\langle \bar{\nu}_f(p_f) | j_\mu^{(\nu)}(0) | \bar{\nu}_i(p_i) \rangle = \overline{u_f}(p_f) \mathcal{C} [\Lambda_\mu^{if}(q)]^T \mathcal{C}^\dagger u_i(p_i), \quad (3.46)$$

where transposition operates in spinor space. Therefore, the effective form-factor matrix in spinor space for antineutrinos is given by

$$\overline{\Lambda}_\mu^{fi}(q) = \mathcal{C} [\Lambda_\mu^{if}(q)]^T \mathcal{C}^\dagger. \quad (3.47)$$

⁵ Here we consider massive neutrinos which are mixed with the three active flavor neutrinos ν_e, ν_μ, ν_τ . This is the case in standard three-neutrino mixing (see Section II) and in its extensions with Dirac sterile neutrinos which mix with the active ones. If there are Dirac sterile neutrinos which are not mixed with the active ones and have non-standard interactions, the CP phases of the corresponding massive neutrinos could be different from that of the standard massive neutrinos. However, since the production and detection of such sterile neutrinos would be very problematic, this case is not interesting in practice.

Using the properties of the charge-conjugation matrix, the expression (3.35) for $\Lambda_\mu^{if}(q)$, and the hermiticity in Eq. (3.36), we obtain the antineutrino form factors

$$\bar{f}_\Omega^{fi} = -f_\Omega^{if} = -(f_\Omega^{fi})^* \quad (\Omega = Q, M, E), \quad (3.48)$$

$$\bar{f}_A^{fi} = f_A^{if} = (f_A^{fi})^*. \quad (3.49)$$

Therefore, in particular the diagonal magnetic and electric moments of neutrinos and antineutrinos, which are real, have the same size with opposite signs, as the charge, if it exists. On the other hand, the real diagonal neutrino and antineutrino anapole moments are equal.

It is interesting to note that the relations in Eqs. (3.48) and (3.49) between neutrino and antineutrino form factors are a consequence of CPT symmetry, which is a fundamental symmetry of local relativistic Quantum Field Theory (see Greenberg (2006)). In order to prove this statement, let us first consider the CPT transformation of the Standard Model electric current $j_\mu(x)$ in Eq. (3.1): using Eq. (A66) we have

$$j_\mu(x) \xrightarrow{\text{CPT}} \mathbf{U}_{\text{CPT}} j_\mu(x) \mathbf{U}_{\text{CPT}}^\dagger = -j_\mu(-x). \quad (3.50)$$

Therefore, the Standard Model electromagnetic interaction Hamiltonian $\mathcal{H}_{\text{em}}^{(\nu)}(x)$ is left invariant by

$$A_\mu(x) \xrightarrow{\text{CPT}} -A_\mu(-x). \quad (3.51)$$

CPT is conserved by the effective neutrino electromagnetic interaction Hamiltonian in Eq. (3.27) if $j_\mu^{(\nu)}(x)$ transforms as $j_\mu(x)$:

$$\text{CPT} \iff \mathbf{U}_{\text{CPT}} j_\mu^{(\nu)}(x) \mathbf{U}_{\text{CPT}}^\dagger = -j_\mu^{(\nu)}(-x). \quad (3.52)$$

In order to find the implications of this relation for the antineutrino matrix element in Eq. (3.45), we need to consider it taking into account the helicities of the initial and final neutrinos, because CPT reverses helicities. Thus, assuming CPT and inserting $\mathbf{U}_{\text{CPT}}^\dagger \mathbf{U}_{\text{CPT}} = 1$ on both sides of $j_\mu^{(\nu)}(0)$, we obtain

$$\begin{aligned} \bar{M}_{fi} &= \langle \bar{\nu}_f(p_f, h_f) | j_\mu^{(\nu)}(0) | \bar{\nu}_i(p_i, h_i) \rangle \\ &= -\langle \bar{\nu}_f(p_f, h_f) | \mathbf{U}_{\text{CPT}}^\dagger j_\mu^{(\nu)}(0) \mathbf{U}_{\text{CPT}} | \bar{\nu}_i(p_i, h_i) \rangle. \end{aligned} \quad (3.53)$$

Now we take into account that the application of \mathbf{U}_{CPT} to a neutrino state transforms it into an antineutrino state. Using the notation and conventions of Giunti and Kim (2007) we have

$$\mathbf{U}_{\text{CPT}} |\bar{\nu}_k(p_k, h_k)\rangle = -\zeta(h) \xi_k^{\text{CPT}*} |\nu_k(p_k, -h_k)\rangle, \quad (3.54)$$

where $\zeta(h)$ is a phase coming from the relation

$$\gamma^5 v^{(-h)}(p) = \zeta(h) u^{(h)}(p), \quad (3.55)$$

and $\zeta(-h) = -\zeta(h)$. For the CPT phases ξ_k^{CPT} , we assume that they are all equal, as we have done for the CP

phases in Eq. (3.40). Then, using Eq. (3.54) and taking into account the antiunitarity of \mathbf{U}_{CPT} , Eq. (3.53) becomes

$$\bar{M}_{fi} = -\zeta(h_f) \zeta^*(h_i) \langle \nu_i(p_i, -h_i) | j_\mu^{(\nu)}(0) | \nu_f(p_f, -h_f) \rangle. \quad (3.56)$$

This is the crucial relation between the neutrino and antineutrino matrix elements which follows from CPT invariance. Using for the neutrino matrix element the expression (3.28) and the relation (3.55), we obtain

$$\bar{M}_{fi} = \overline{v_i^{(h_i)}}(p_i) \gamma^5 \Lambda_\mu^{if}(-q) \gamma^5 v_f^{(h_f)}(p_f). \quad (3.57)$$

Taking into account the form-factor expression of $\Lambda_\mu^{fi}(q)$ in Eq. (3.35), we have $\gamma^5 \Lambda_\mu^{if}(-q) \gamma^5 = -\Lambda_\mu^{if}(q)$, which leads to

$$\bar{M}_{fi} = -\overline{v_i^{(h_i)}}(p_i) \Lambda_\mu^{if}(q) v_f^{(h_f)}(p_f). \quad (3.58)$$

This expression for the antineutrino matrix element coincides with Eq. (3.45) and implies the relations (3.48) and (3.49) for the form factors.

Thus, we obtained the expression (3.45) for the antineutrino matrix element in a complicated way, assuming only CPT invariance and the expression (3.28) for the neutrino matrix element. This result is a tautology in the theoretical framework in which we are working, because CPT is a fundamental symmetry of any local relativistic Quantum Field Theory (see Greenberg (2006)). However, in some theories beyond the Standard Model small CPT violations can exist (see Tsukerman (2010)), which may be revealed by finding violations of the equalities in Eqs. (3.48) and (3.49).

B. Majorana neutrinos

A Majorana neutrino is a neutral spin 1/2 particle which coincides with its antiparticle. The four degrees of freedom of a Dirac field (two helicities and two particle-antiparticle) are reduced to two (two helicities) by the Majorana constraint in Eq. (2.25). Since a Majorana field has half the degrees of freedom of a Dirac field, it is possible that its electromagnetic properties are reduced. From the relations (3.48) and (3.49) between neutrino and antineutrino form factors in the Dirac case, we can infer that in the Majorana case the charge, magnetic and electric form-factor matrices are antisymmetric and the anapole form-factor matrix is symmetric. In order to confirm this deduction, let us calculate the neutrino matrix element corresponding to the effective electromagnetic vertex in Fig. 5, with the effective interaction Hamiltonian in Eq. (3.27), which takes into account possible transitions between two different initial and final massive Majorana neutrinos ν_i and ν_f . Using the Fourier

expansion (A57) for the neutrino Majorana fields we obtain

$$\begin{aligned} \langle \nu_f(p_f) | j_\mu^{(\nu)}(0) | \nu_i(p_i) \rangle &= \overline{u}_f(p_f) \Lambda_\mu^{fi}(p_f, p_i) u_i(p_i) \\ &\quad - \overline{v}_i(p_i) \Lambda_\mu^{if}(p_f, p_i) v_f(p_f). \end{aligned} \quad (3.59)$$

Using Eq. (A45), we can write it as

$$\overline{u}_f(p_f) \{ \Lambda_\mu^{fi}(p_f, p_i) + \mathcal{C} [\Lambda_\mu^{if}(p_f, p_i)]^T \mathcal{C}^\dagger \} u_i(p_i), \quad (3.60)$$

where transposition operates in spinor space. Therefore the effective form-factor matrix in spinor space for Majorana neutrinos is given by

$$\Lambda_\mu^{Mfi}(p_f, p_i) = \Lambda_\mu^{fi}(p_f, p_i) + \mathcal{C} [\Lambda_\mu^{if}(p_f, p_i)]^T \mathcal{C}^\dagger. \quad (3.61)$$

As in the case of Dirac neutrinos, $\Lambda_\mu^{fi}(p_f, p_i)$ depends only on $q = p_f - p_i$ and can be expressed in terms of six Lorentz-invariant form factors according to Eq. (3.29). Hence, we can write the $N \times N$ matrix $\Lambda_\mu^M(p_f, p_i)$ in the space of massive Majorana neutrinos as

$$\begin{aligned} \Lambda_\mu^M(q) &= \mathbb{f}_1^M(q^2) q_\mu + \mathbb{f}_2^M(q^2) q_\mu \gamma_5 + \mathbb{f}_3^M(q^2) \gamma_\mu \\ &\quad + \mathbb{f}_4^M(q^2) \gamma_\mu \gamma_5 + \mathbb{f}_5^M(q^2) \sigma_{\mu\nu} q^\nu \\ &\quad + \mathbb{f}_6^M(q^2) \epsilon_{\mu\nu\rho\gamma} q^\nu \sigma^{\rho\gamma}, \end{aligned} \quad (3.62)$$

with

$$\mathbb{f}_k^M = \mathbb{f}_k + \mathbb{f}_k^T \Rightarrow \mathbb{f}_k^M = (\mathbb{f}_k^M)^T \quad \text{for } k = 1, 2, 4, \quad (3.63)$$

$$\mathbb{f}_k^M = \mathbb{f}_k - \mathbb{f}_k^T \Rightarrow \mathbb{f}_k^M = -(\mathbb{f}_k^M)^T \quad \text{for } k = 3, 5, 6. \quad (3.64)$$

Now we can follow the discussion in Section III.A for Dirac neutrinos taking into account the additional constraints (3.63) and (3.64) for Majorana neutrinos. The hermiticity of $j_\mu^{(\nu)}$ and current conservation lead to an expression similar to that in Eq. (3.37):

$$\begin{aligned} \Lambda_\mu^M(q) &= (\gamma_\mu - q_\mu \not{q}/q^2) [\mathbb{f}_Q^M(q^2) + \mathbb{f}_A^M(q^2) q^2 \gamma_5] \\ &\quad - i \sigma_{\mu\nu} q^\nu [\mathbb{f}_M^M(q^2) + i \mathbb{f}_E^M(q^2) \gamma_5], \end{aligned} \quad (3.65)$$

with $\mathbb{f}_Q^M = \mathbb{f}_3^M$, $\mathbb{f}_M^M = i \mathbb{f}_5^M$, $\mathbb{f}_E^M = -2i \mathbb{f}_6^M$ and $\mathbb{f}_A^M = -\mathbb{f}_2^M/(m_f + m_i)$. For the Hermitian $N \times N$ form-factor matrices in the space of massive neutrinos,

$$\mathbb{f}_\Omega^M = (\mathbb{f}_\Omega^M)^\dagger \quad (\Omega = Q, M, E, A), \quad (3.66)$$

the Majorana constraints (3.63) and (3.64) imply that

$$\mathbb{f}_\Omega^M = -(\mathbb{f}_\Omega^M)^T \quad (\Omega = Q, M, E), \quad (3.67)$$

$$\mathbb{f}_A^M = (\mathbb{f}_A^M)^T. \quad (3.68)$$

These relations confirm the expectation discussed above that for Majorana neutrinos the charge, magnetic and electric form-factor matrices are antisymmetric and the anapole form-factor matrix is symmetric.

Since \mathbb{f}_Q^M , \mathbb{f}_M^M and \mathbb{f}_E^M are antisymmetric, a Majorana neutrino does not have diagonal charge and dipole magnetic and electric form factors (Radicati and Touschek,

1957; Case, 1957). It can only have a diagonal anapole form factor. On the other hand, Majorana neutrinos can have as many off-diagonal (transition) form-factors as Dirac neutrinos.

Since the form-factor matrices are Hermitian as in the Dirac case, \mathbb{f}_Q^M , \mathbb{f}_M^M and \mathbb{f}_E^M are imaginary, whereas \mathbb{f}_A^M is real:

$$\mathbb{f}_\Omega^M = -(\mathbb{f}_\Omega^M)^* \quad (\Omega = Q, M, E), \quad (3.69)$$

$$\mathbb{f}_A^M = (\mathbb{f}_A^M)^*. \quad (3.70)$$

Taking into account these properties, in the standard case of three-neutrino mixing the charge, magnetic and electric Majorana form factors can be written as

$$\mathbb{f}_\Omega^{Mfi}(q^2) = i \sum_{j=1}^3 \epsilon^{fij} \tilde{\mathbb{f}}_\Omega^{Mj}(q^2), \quad (3.71)$$

for $\Omega = Q, M, E$, in terms of three vectors of real form factors

$$(\tilde{\mathbb{f}}_\Omega^{M1}, \tilde{\mathbb{f}}_\Omega^{M2}, \tilde{\mathbb{f}}_\Omega^{M3}) = -i(\tilde{\mathbb{f}}_\Omega^{M23}, \tilde{\mathbb{f}}_\Omega^{M31}, \tilde{\mathbb{f}}_\Omega^{M12}). \quad (3.72)$$

Considering now CP invariance, the case of Majorana neutrinos is rather different from that of Dirac neutrinos, because the CP phases of Majorana neutrinos are constrained by the CP invariance of the Majorana mass term. In order to prove this statement, let us first notice that since a massive Majorana neutrino field ν_k is constrained by the Majorana relation in Eq. (2.25), only the parity transformation part is effective in a CP transformation. Indeed, from Eqs. (2.25) and (A64) we obtain

$$\mathcal{U}_{\text{CP}} \nu_k(x) \mathcal{U}_{\text{CP}}^\dagger = \xi_k^{\text{CP}} \gamma^0 \nu_k(x_P). \quad (3.73)$$

Considering the Majorana mass term in Eq. (2.23), we have

$$\mathcal{U}_{\text{CP}} \nu_k^T \mathcal{C}^\dagger \nu_k \mathcal{U}_{\text{CP}}^\dagger = -\xi_k^{\text{CP}2} \nu_k^T \mathcal{C}^\dagger \nu_k. \quad (3.74)$$

Therefore,

$$\text{CP} \iff \xi_k^{\text{CP}} = \eta_k i, \quad (3.75)$$

with $\eta_k = \pm 1$. These CP signs can be different for the different massive neutrinos, even if they all take part to the standard charged-current weak interactions through neutrino mixing, because they can be compensated by the Majorana CP phases in the mixing matrix (see Giunti and Kim (2007)). Therefore, from Eq. (3.40) we have

$$\Lambda_\mu^{Mfi}(q) \xrightarrow{\text{CP}} \eta_f \eta_i \gamma^0 \mathcal{C} [\Lambda_\mu^{Mif}(q_P)]^T \mathcal{C}^\dagger \gamma^0. \quad (3.76)$$

Imposing a CP constraint analogous to that in Eq. (3.23), we obtain

$$\text{CP} \iff \begin{cases} \mathbb{f}_\Omega^{Mfi} = \eta_f \eta_i \mathbb{f}_\Omega^{Mfi} = \eta_f \eta_i (\mathbb{f}_\Omega^{Mfi})^*, \\ \mathbb{f}_E^{Mfi} = -\eta_f \eta_i \mathbb{f}_E^{Mfi} = -\eta_f \eta_i (\mathbb{f}_E^{Mfi})^*, \end{cases} \quad (3.77)$$

with $\Omega = Q, M, A$. Taking into account the constraints (3.69) and (3.70), we have two cases:

$$\text{CP} \quad \text{and} \quad \eta_f = \eta_i \iff \mathbb{f}_Q^{Mfi} = \mathbb{f}_M^{Mfi} = 0, \quad (3.78)$$

and

$$\text{CP} \quad \text{and} \quad \eta_f = -\eta_i \iff \mathbb{f}_E^{Mfi} = \mathbb{f}_A^{Mfi} = 0. \quad (3.79)$$

Therefore, if CP is conserved two massive Majorana neutrinos can have either a transition electric form factor or a transition magnetic form factor, but not both, and the transition electric form factor can exist only together with a transition anapole form factor, whereas the transition magnetic form factor can exist only together with a transition charge form factor. In the diagonal case $f = i$, Eq. (3.78) does not give any constraint, because only diagonal anapole form factors are allowed for Majorana neutrinos.

Let us finally consider the CPT symmetry. Following the method used at the end of the previous Section III.A for Dirac neutrinos and taking into account the particle-antiparticle equality of Majorana neutrinos, one can show that the relations (3.67) and (3.68) are a consequence of CPT symmetry (Nieves, 1982; Kayser, 1982, 1984). Therefore, in particular the existence of diagonal magnetic or electric moments of Majorana neutrinos would be a signal of CPT violation.

C. Massless Weyl neutrinos

In Section II we have seen that neutrinos are known to be massive and mixed. However, it is interesting to study the electromagnetic properties of neutrinos in the Standard Model, where they are described by two-component massless left-handed Weyl spinors. In this case, taking into account that there is no mixing, the effective electromagnetic interaction Hamiltonian is

$$\mathcal{H}_{\text{em}}^{(\nu)}(x) = \sum_{\alpha, \beta=e, \mu, \tau} \bar{\nu}_\alpha(x) \Lambda_\mu^{\alpha\beta} \nu_\beta(x) A^\mu(x). \quad (3.80)$$

Since neutrinos are strictly left-handed, the effective electromagnetic vertex in Fig. 5 is given by the matrix element

$$\langle \nu_\alpha(p_\alpha, -) | j_\mu^{(\nu)}(0) | \nu_\beta(p_\beta, -) \rangle = \overline{u_\alpha^{(-)}}(p_\alpha) \Lambda_\mu^{fi}(q) u_\beta^{(-)}(p_\beta), \quad (3.81)$$

with $q = p_\beta - p_\alpha$. Since for massless neutrinos Eq. (C6) leads to the equality

$$\gamma^5 u^{(-)}(p) = -u^{(-)}(p), \quad (3.82)$$

we can reduce the general expression of Λ_μ in Eq. (3.37) to (Bernstein *et al.*, 1963)

$$\Lambda_\mu(q) = (\gamma_\mu - q_\mu \not{q}/q^2) \mathbb{f}(q^2), \quad (3.83)$$

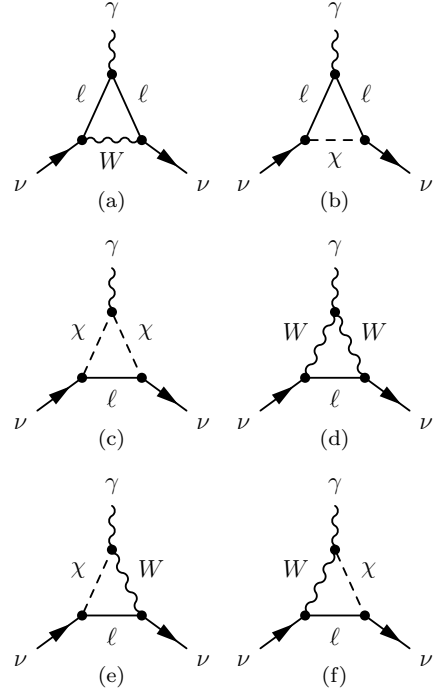


FIG. 6 Contributions to the neutrino vertex function from proper vertices (χ is the unphysical would-be charged scalar boson; the corresponding Feynman rules necessary for the massive neutrino electromagnetic vertex calculations can be found in Dvornikov and Studenikin (2004a,b)).

with

$$\mathbb{f}(q^2) = \mathbb{f}_Q(q^2) - \mathbb{f}_A(q^2)q^2. \quad (3.84)$$

Therefore, massless left-handed Weyl neutrinos have only one type of form factor given by the sum of the charge form factor and the anapole form factor multiplied by q^2 .

It is important that massless left-handed Weyl neutrinos cannot have diagonal or off-diagonal electric or magnetic dipole moments. The physical reason is that in the case of massless neutrinos electric and magnetic dipole moments correspond to terms proportional to $\sigma_{\mu\nu}$, which flip helicity, as explained in Appendix C, but the helicity flip of a massless left-handed Weyl neutrino is not possible if the corresponding right-handed state does not exist.

D. Form factors in gauge models

The form factors at $q^2 = 0$ describe the interactions of neutrinos with real photons and determine the interactions of neutrinos with classical electromagnetic fields in the static limit ($q = 0$; see Section VI). Hence, the form factors at $q^2 = 0$ are quantities that are directly measurable, at least in principle. Then, it follows that in any

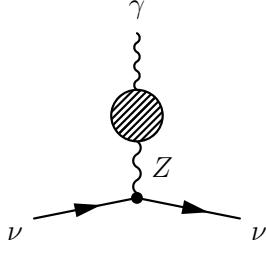


FIG. 7 Contributions to the neutrino vertex function of γ - Z self-energy diagrams.

realistic theoretical model the values of the form factors at $q^2 = 0$ must be invariant under gauge transformations.

On the other hand, in non-Abelian gauge theories the form factors at $q^2 \neq 0$ can be non-invariant under gauge transformations, because in general the off-shell photon propagator can be gauge-dependent. In this case, the one-photon approximation is not enough to obtain measurable quantities, which are given by gauge-invariant sums of higher-order diagrams (see Bardeen *et al.* (1972)).

Considering the full set of one-loop Feynman diagrams contributing to the Dirac massive neutrino electromagnetic vertex function in the framework of the Standard Model supplied with the SU(2)-singlet right-handed neutrino in the general R_ξ gauge (Dvornikov and Studenikin, 2004a,b), the vertex function $\Lambda_\mu(q)$, in the one-loop approximation, contains contributions given by two types of diagrams: the proper vertices (Fig. 6) and the γ - Z self-energy diagrams (Fig. 7).

A direct calculation of the massive neutrino electromagnetic vertex function, taking into account all the diagrams (Fig. 6 and Fig. 8), reveals that each of the Feynman diagrams gives nonzero contribution to the term proportional to $\gamma_\mu \gamma_5$ (Dvornikov and Studenikin, 2004a,b). These contributions are not vanishing even at $q^2 = 0$. Therefore, in addition to the usual four terms in Eq. (3.18) an extra term proportional to $\gamma_\mu \gamma_5$ appears and a corresponding additional form factor $\mathbb{f}_5(q^2)$ must be introduced. This problem is related to the decomposition of the massive neutrino electromagnetic vertex function. The calculation of the contributions of the proper vertex diagrams (Fig. 6) and γ - Z self-energy diagrams (Fig. 7) for arbitrary gauge fixing parameter $\alpha = 1/\xi$ and arbitrary mass parameter $a = m_l^2/m_W^2$ shows that at least in the zeroth and first orders of the expansion over the small neutrino mass parameter $b = (m_\nu/m_W)^2$ the corresponding “charge” $\mathbb{f}_5(q^2 = 0)$ is zero. The cancellation of contributions from the proper vertex and self-energy diagrams to the form factor $\mathbb{f}_5(q^2)$ at $q^2 \neq 0$,

$$\mathbb{f}_5(q^2) = \mathbb{f}_5^{(\gamma-Z)}(q^2) + \mathbb{f}_5^{(\text{prop. vert.})}(q^2) = 0, \quad (3.85)$$

was also shown by Dvornikov and Studenikin (2004a,b) for arbitrary mass parameters a and b in the ‘t Hooft-Feynman gauge $\alpha = 1$.

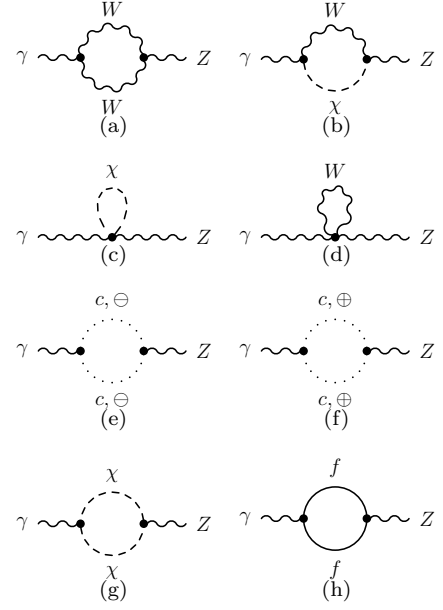


FIG. 8 γ - Z self-energy diagrams. f denotes the electron, muon, τ -lepton and u , c , t , d , s and b quarks (the charge of ghosts is indicated by the symbols \oplus and \ominus).

For a massive Dirac neutrino, by performing a direct calculations of the complete set of one-loop diagrams one can find that the neutrino vertex function consists of only three electromagnetic form factors (in the case of a model with CP conservation) (Dvornikov and Studenikin, 2004a,b). Closed integral expressions are found for electric, magnetic, and anapole form factors of a massive neutrino. On this basis, the electric charge (the value of the electric form factor at zero momentum transfer), magnetic moment, and anapole moment of a *massive* neutrino have been derived. It has been shown by means of direct calculations for the case of a massive neutrino that the electric charge is independent of the gauge parameters and is equal to zero, the magnetic moment is finite and does not depend on the choice of gauge.

IV. MAGNETIC AND ELECTRIC DIPOLE MOMENTS

The magnetic and electric dipole moments are theoretically the most well-studied and understood electromagnetic properties of neutrinos. They also attract some interest from experimentalists, although the magnetic moments of Dirac neutrinos in the simplest extension of the Standard Model with the addition of right-handed neutrinos are proportional to the corresponding neutrino mass and therefore they are many orders of magnitude smaller than the present experimental limits obtained in terrestrial experiments.

In Subsection IV.A we discuss this prediction for Dirac

neutrinos and in Subsection IV.B we present the predictions for the transition magnetic moments of Majorana neutrinos in minimal extensions of the Standard Model. In Subsection IV.C we discuss the observable effects of electric and magnetic dipole moments in neutrino-electron elastic scattering and in Subsection IV.D we review the derivation of the effective dipole moments in scattering experiments. In Subsection IV.E we present the most relevant experimental limits on the values of the effective dipole moments and in Subsection IV.F we conclude with some considerations on the theoretical possibilities to have large magnetic moments.

A. Theoretical predictions for Dirac neutrinos

The first calculations of the dipole moments of Dirac neutrinos in the minimal extension of the Standard Model with right-handed neutrinos were performed in Marciano and Sanda (1977); Lee and Shrock (1977); Fujikawa and Shrock (1980); Petcov (1977); Pal and Wolfenstein (1982); Shrock (1982) by evaluating the radiative diagrams shown in Fig. 6. The explicit evaluation of the one-loop contributions to the neutrino dipole moments in the leading approximation over the small parameters $b_k = m_k^2/m_W^2$ (where m_k are the neutrino masses, with $k = 1, 2, 3$), that in addition exactly accounts for the dependence on the small parameters $a_l = m_l^2/m_W^2$ (with $l = e, \mu, \tau$), yields (Petcov, 1977; Pal and Wolfenstein, 1982; Shrock, 1982; Bilenky and Petcov, 1987; Mohapatra and Pal, 2004)

$$\left\{ \begin{array}{c} \mu_{kj}^D \\ \epsilon_{kj}^D \end{array} \right\} = \frac{eG_F}{8\sqrt{2}\pi^2} (m_k \pm m_j) \sum_{l=e,\mu,\tau} f(a_l) U_{lk}^* U_{lj}, \quad (4.1)$$

where the subscript “D” indicate Dirac neutrinos and

$$f(a_l) = \frac{3}{4} \left[1 + \frac{1}{1-a_l} - \frac{2a_l}{(1-a_l)^2} - \frac{2a_l^2 \ln a_l}{(1-a_l)^3} \right]. \quad (4.2)$$

Since all the charged lepton parameters a_l are small, we can approximate

$$f(a_l) \simeq \frac{3}{2} \left(1 - \frac{a_l}{2} \right), \quad (4.3)$$

and we obtain

$$\left\{ \begin{array}{c} \mu_{kj}^D \\ \epsilon_{kj}^D \end{array} \right\} \simeq \frac{3eG_F}{16\sqrt{2}\pi^2} (m_k \pm m_j) \times \left(\delta_{kj} - \frac{1}{2} \sum_{l=e,\mu,\tau} U_{lk}^* U_{lj} \frac{m_l^2}{m_W^2} \right). \quad (4.4)$$

It is clear that in this model there are no diagonal electric dipole moments ($\epsilon_{kk}^D = 0$). The diagonal magnetic moments are given by

$$\mu_{kk}^D \simeq \frac{3eG_F m_k}{8\sqrt{2}\pi^2} \left(1 - \frac{1}{2} \sum_{l=e,\mu,\tau} |U_{lk}|^2 \frac{m_l^2}{m_W^2} \right). \quad (4.5)$$

This result exhibits the following important features.

Each diagonal magnetic moments is proportional to the corresponding neutrino mass and vanishes in the massless limit, even if in the extension of the Standard Model under consideration there are right-handed neutrinos. This case is different from that of massless Weyl neutrinos discussed in Section III.C, in which all electric and magnetic, diagonal and off-diagonal dipole moments are forbidden by the absence of right-handed states. In this case we have both spinors $u^{(-)}(p)$ and $u^{(+)}(p)$. As shown in Appendix C, in the massless limit helicity equals chirality, because $\gamma^5 u^{(\pm)}(p) = \pm u^{(\pm)}(p)$. Since $\bar{u}^{(\pm)}(p) \sigma^{\mu\nu} u^{(\pm)}(p) = 0$ and $\bar{u}^{(\pm)}(p) \sigma^{\mu\nu} u^{(\mp)}(p) \neq 0$, the existence of a magnetic moment corresponds to the existence of an helicity and chirality flipping interaction with the electromagnetic field. However, in the minimal extension of the Standard Model with right-handed neutrinos a magnetic moment is generated by the radiative diagrams in Fig. 6, which cannot flip chirality, because the weak interaction vertices in the diagrams in Fig. 6 involve only left-handed neutrinos. As discussed briefly at the end of this Section, in order to generate a magnetic moment of massless neutrinos it is necessary to extend the model by introducing, for example, right-handed charged currents.

At the leading order in the small ratio m_l^2/m_W^2 , the diagonal magnetic moments are independent of the neutrino mixing matrix and of the values of the charged lepton masses. Their numerical values are given by

$$\mu_{kk}^D \simeq 3.2 \times 10^{-19} \left(\frac{m_k}{\text{eV}} \right) \mu_B. \quad (4.6)$$

Taking into account the existing constraint of the order of 1 eV on the neutrino masses (see Section II.E), these values are several orders of magnitude smaller than the present experimental limits, which are discussed in Section IV.E.

Let us consider now the neutrino transition dipole moments, which are given by Eqs. (4.1) and (4.4) for $k \neq j$. Considering only the leading term $f(a_l) \simeq 3/2$ in the expansion (4.3), one gets vanishing transition dipole moments, because of the unitarity relation

$$\sum_l U_{lk}^* U_{lj} = \delta_{kj}. \quad (4.7)$$

Therefore, the first nonvanishing contribution comes from the second term in the expansion (4.3) of $f(a_l)$, which contains the additional small factor $a_l = m_l^2/m_W^2$:

$$\left\{ \begin{array}{c} \mu_{kj}^D \\ \epsilon_{kj}^D \end{array} \right\} \simeq -\frac{3eG_F}{32\sqrt{2}\pi^2} (m_k \pm m_j) \sum_{l=e,\mu,\tau} U_{lk}^* U_{lj} \frac{m_l^2}{m_W^2}, \quad (4.8)$$

for $k \neq j$. Thus, the transition magnetic moment μ_{kj}^D is suppressed with respect to the largest of the diagonal magnetic moments of ν_k and ν_j , which are given by

Eq. (4.5). This suppression is called “GIM mechanism”, in analogy with the suppression of flavor-changing neutral currents in hadronic processes discovered by Glashow *et al.* (1970). Numerically, the transition dipole moments are given by

$$\left\{ \begin{array}{l} \mu_{kj}^D \\ \epsilon_{kj}^D \end{array} \right\} \simeq -3.9 \times 10^{-23} \mu_B \left(\frac{m_k \pm m_j}{\text{eV}} \right) \times \sum_{l=e,\mu,\tau} U_{lk}^* U_{lj} \left(\frac{m_l}{m_\tau} \right)^2. \quad (4.9)$$

Hence, the suppression of μ_{kj}^D with respect to the numerical values of the largest of the diagonal magnetic moments of ν_k and ν_j , which are given by Eq. (4.6), is at least a factor of the order of 10^{-4} . The transition electric moments are even smaller than the transition magnetic moment because of the mass difference, but they are the only electric moments in the extension of the Standard Model under consideration.

In recent studies, the value of the diagonal magnetic moment of a massive Dirac neutrino was calculated in the one-loop approximation in the extended Standard Model with right-handed neutrinos, accounting for the dependence on the neutrino mass parameter $b_k = m_k^2/m_W^2$ (Cabral-Rosetti *et al.*, 2000) and accounting for the exact dependence on both mass parameters b_k and $a_l = m_l^2/m_W^2$ (Dvornikov and Studenikin, 2004a,b). The calculations of the neutrino magnetic moment which take into account exactly the dependence on the masses of all particles can be useful in the case of a heavy neutrino with a mass comparable or even exceeding the values of the masses of other known particles. Note that the LEP data require that the number of light neutrinos coupled to the Z boson is three (Schael *et al.*, 2006). Therefore, any additional active neutrino must be heavier than $m_Z/2$. This possibility is not excluded by current data (see Cetin *et al.* (2011)).

For a heavy neutrino with mass m_k much larger than the charged lepton masses but smaller than the W -boson mass ($2 \text{ GeV} \ll m_k \ll 80 \text{ GeV}$), Dvornikov and Studenikin (2004a,b) obtained the diagonal magnetic moment

$$\mu_{kk} \simeq \frac{3eG_F}{8\pi^2\sqrt{2}} m_k \left(1 + \frac{5}{18} b_k \right), \quad (4.10)$$

whereas for a heavy neutrino with mass m_k much larger than the W -boson mass, they got

$$\mu_{kk} \simeq \frac{eG_F}{8\pi^2\sqrt{2}} m_k. \quad (4.11)$$

Note that in both cases the Dirac neutrino magnetic moment is proportional to the neutrino mass. This is an expected result, because the calculations have been performed within the extended Standard Model with right-handed neutrinos.

At this point, a question arises: “Is a neutrino magnetic moment always proportional to the neutrino mass?”. The answer is “No”. For example, much larger values of the Dirac neutrino magnetic moment can be obtained in $SU(2)_L \times SU(2)_R \times U(1)$ left-right symmetric models with direct right-handed neutrino interactions (see, for instance, Kim (1976); Marciano and Sanda (1977); Czakon *et al.* (1999); Beg *et al.* (1978)). The massive gauge bosons states W_1 and W_2 have, respectively, predominant left-handed and right-handed coupling, since

$$W_1 = W_L \cos \xi - W_R \sin \xi, \quad (4.12)$$

$$W_2 = W_L \sin \xi + W_R \cos \xi, \quad (4.13)$$

where ξ is a small mixing angle and the fields W_L and W_R have pure $V \pm A$ interactions. The magnetic moment of a neutrino ν_l calculated in this model, neglecting neutrino mixing, is

$$\mu_{\nu_l} = \frac{eG_F}{2\sqrt{2}\pi^2} \left[m_l \left(1 - \frac{m_{W_1}^2}{m_{W_2}^2} \right) \sin 2\xi + \frac{3}{4} m_{\nu_l} \left(1 + \frac{m_{W_1}^2}{m_{W_2}^2} \right) \right]. \quad (4.14)$$

where the term proportional to the charged lepton mass m_l is due to the left-right mixing. This term can exceed the second term in Eq. (4.14), which is proportional to the neutrino mass m_{ν_l} .

B. Theoretical predictions for Majorana neutrinos

Also Majorana neutrinos can have nonvanishing transition magnetic and electric moments, as discussed in Section III.B. The simplest models with Majorana neutrinos can be obtained by extending the Standard Model with the addition of a $SU(2)_L$ Higgs triplet (Gelmini and Roncadelli, 1981) or with the addition of right-handed neutrinos and a $SU(2)_L$ Higgs singlet (Chikashige *et al.*, 1980) (see Mohapatra and Pal (2004)). Neglecting the model-dependent Feynman diagrams which depend on the details of the scalar sector, the Majorana magnetic and electric transition moments are given by (Shrock, 1982)

$$\mu_{kj}^M \simeq -\frac{3eG_F}{16\sqrt{2}\pi^2} (m_k + m_j) \sum_{l=e,\mu,\tau} \text{Im}[U_{lk}^* U_{lj}] \frac{m_l^2}{m_W^2}, \quad (4.15)$$

$$\epsilon_{kj}^M \simeq -\frac{3eG_F}{16\sqrt{2}\pi^2} (m_k - m_j) \sum_{l=e,\mu,\tau} \text{Re}[U_{lk}^* U_{lj}] \frac{m_l^2}{m_W^2}. \quad (4.16)$$

Apart from the increase by a factor of 2 of the first coefficient with respect to the Dirac case in Eq. (4.8), it is

difficult to compare the expressions of the Dirac and Majorana dipole moments, because the mixing matrices are different in the two cases, due to the possible presence of additional phases in the Majorana case (see Eq. (2.27)). In any case, it is clear that also the Majorana transition dipole moments are suppressed by the GIM mechanism. However, the model-dependent contributions of the scalar sector can enhance the Majorana transition dipole moments (see Pal and Wolfenstein (1982); Barr *et al.* (1990); Pal (1991)).

If CP is conserved, we must distinguish the two cases in which ν_k and ν_j have the same or opposite CP phases, as explained in Section III.B. It can be shown (see Giunti and Kim (2007)) that if CP is conserved the elements of the mixing matrix can be written as

$$U_{lk} = \mathcal{O}_{lk} e^{i\lambda_k}, \quad (4.17)$$

where \mathcal{O} is a real orthogonal matrix (e.g. U^D in Eq. (2.28) with $\delta_{13} = 0, \pi$) and the Majorana CP phases λ_k such that

$$e^{-2i(\lambda_k - \lambda_j)} = \eta_k / \eta_j. \quad (4.18)$$

Here $\eta_k = \pm 1$ is the sign of the CP phase in Eq. (3.75) of the massive Majorana neutrino ν_k . Then, we have

$$U_{lk}^* U_{lj} = \mathcal{O}_{lk} \mathcal{O}_{lj} e^{-i(\lambda_k - \lambda_j)} = \mathcal{O}_{lk} \mathcal{O}_{lj} \sqrt{\eta_k / \eta_j}. \quad (4.19)$$

Then, if ν_k and ν_j have the same CP phase ($\eta_k = \eta_j$), the products $U_{lk}^* U_{lj} = \mathcal{O}_{lk} \mathcal{O}_{lj}$ are real and the dipole moments are given by (Schechter and Valle, 1981; Pal and Wolfenstein, 1982)

$$\mu_{kj}^M = 0 \quad \text{and} \quad \epsilon_{kj}^M = 2\epsilon_{kj}^D, \quad (4.20)$$

with ϵ_{kj}^D and μ_{kj}^D given by Eq. (4.1). On the other hand, if ν_k and ν_j have opposite CP phases ($\eta_k = -\eta_j$), the products $U_{lk}^* U_{lj} = i\mathcal{O}_{lk} \mathcal{O}_{lj}$ are imaginary and the dipole moments are given by (Schechter and Valle, 1981; Pal and Wolfenstein, 1982)

$$\mu_{kj}^M = 2\mu_{kj}^D \quad \text{and} \quad \epsilon_{kj}^D = 0. \quad (4.21)$$

The vanishing of μ_{kj}^M in the first case and the vanishing of ϵ_{kj}^D in the second case are consistent with the general results in Eqs. (3.78) and (3.79).

C. Neutrino-electron elastic scattering

The most sensitive and widely used method for the experimental investigation of the neutrino magnetic moment is provided by direct laboratory measurements of low-energy elastic scattering of neutrinos and antineutrinos with electrons in reactor, accelerator and solar experiments. Detailed descriptions of several experiments can be found in Wong and Li (2005); Beda *et al.* (2007).

Extensive experimental studies of the neutrino magnetic moment, performed during many years, are stimulated by the hope to observe a value much larger than the prediction in Eq. (4.6) of the minimally extended Standard Model with right-handed neutrinos. It would be a clear indication of new physics beyond the extended Standard Model. For example, the effective magnetic moment in $\bar{\nu}_e e$ elastic scattering in a class of extra-dimension models can be as large as about $10^{-10} \mu_B$ (Mohapatra *et al.*, 2004). Future higher precision reactor experiments can therefore be used to provide new constraints on large extra-dimensions.

The possibility for neutrino-electron elastic scattering due to neutrino magnetic moment was first considered in Carlson and Oppenheimer (1932) and the cross section of this process was calculated in Bethe (1935) (for related short historical notes see Kyuldjiev (1984)). Here we would like to recall the paper by Domogatsky and Nadezhin (1970), where the cross section of Bethe (1935) was corrected and the antineutrino-electron cross section was considered in the context of the earlier experiments with reactor antineutrinos of Cowan *et al.* (1954); Cowan and Reines (1957), which were aimed to reveal the effects of the neutrino magnetic moment. Discussions on the derivation of the cross section and on the optimal conditions for bounding the neutrino magnetic moment, as well as a collection of cross section formulae for elastic scattering of neutrinos (antineutrinos) on electrons, nucleons, and nuclei can be found in Kyuldjiev (1984); Vogel and Engel (1989).

Let us consider the elastic scattering

$$\bar{\nu}_\ell + e^- \rightarrow \bar{\nu}_\ell + e^- \quad (4.22)$$

of a neutrino or antineutrino with flavor $\ell = e, \mu, \tau$ and energy E_ν with an electron at rest in the laboratory frame. There are two observables: the kinetic energy T_e of the recoil electron and the recoil angle χ with respect to the neutrino beam, which are related by

$$\cos \chi = \frac{E_\nu + m_e}{E_\nu} \left[\frac{T_e}{T_e + 2m_e} \right]^{1/2}. \quad (4.23)$$

The electron kinetic energy is constrained from the energy-momentum conservation by

$$T_e \leq \frac{2E_\nu^2}{2E_\nu + m_e}. \quad (4.24)$$

Since, in the ultrarelativistic limit, the neutrino magnetic moment interaction changes the neutrino helicity and the Standard Model weak interaction conserves the neutrino helicity (see Appendix C), the two contributions add incoherently in the cross section⁶ which can be writ-

⁶ The small interference term due to neutrino masses has been derived by Grimus and Stockinger (1998).

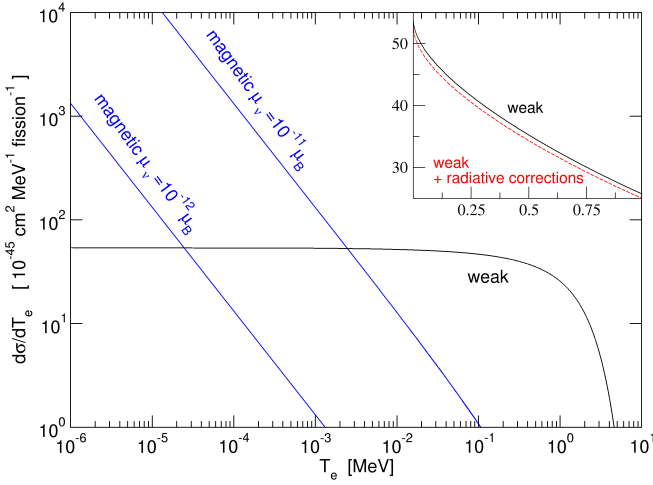


FIG. 9 Standard Model weak and magnetic moment electromagnetic contributions to the differential cross section of neutrino-electron scattering averaged over the antineutrino spectrum of fissioning ^{235}U (Balantekin and Vassh, 2013). The inset plot is the weak correction on the linear scale both with (dashed line) and without (solid line) radiative corrections (Sarantakos *et al.*, 1983).

ten as (Vogel and Engel, 1989),

$$\frac{d\sigma_{\nu_\ell e^-}}{dT_e} = \left(\frac{d\sigma_{\nu_\ell e^-}}{dT_e} \right)_{\text{SM}} + \left(\frac{d\sigma_{\nu_\ell e^-}}{dT_e} \right)_{\text{mag}}. \quad (4.25)$$

The weak-interaction cross section is given by

$$\begin{aligned} \left(\frac{d\sigma_{\nu_\ell e^-}}{dT_e} \right)_{\text{SM}} = & \frac{G_F^2 m_e}{2\pi} \left\{ (g_V^{\nu_\ell} + g_A^{\nu_\ell})^2 \right. \\ & + (g_V^{\nu_\ell} - g_A^{\nu_\ell})^2 \left(1 - \frac{T_e}{E_\nu} \right)^2 \\ & \left. + [(g_A^{\nu_\ell})^2 - (g_V^{\nu_\ell})^2] \frac{m_e T_e}{E_\nu^2} \right\}, \quad (4.26) \end{aligned}$$

with the standard coupling constants g_V and g_A given by

$$g_V^{\nu_e} = 2 \sin^2 \theta_W + 1/2, \quad g_A^{\nu_e} = 1/2, \quad (4.27)$$

$$g_V^{\nu_{\mu,\tau}} = 2 \sin^2 \theta_W - 1/2, \quad g_A^{\nu_{\mu,\tau}} = -1/2. \quad (4.28)$$

For antineutrinos one must substitute $g_A \rightarrow -g_A$.

The neutrino magnetic-moment contribution to the cross section is given by (Vogel and Engel, 1989)

$$\left(\frac{d\sigma_{\nu_\ell e^-}}{dT_e} \right)_{\text{mag}} = \frac{\pi \alpha^2}{m_e^2} \left(\frac{1}{T_e} - \frac{1}{E_\nu} \right) \left(\frac{\mu_{\nu_\ell}}{\mu_B} \right)^2, \quad (4.29)$$

where μ_{ν_ℓ} is the effective magnetic moment discussed in the following Subsection IV.D. It is called traditionally “magnetic moment”, but it receives equal contributions from both the electric and magnetic dipole moments.

The two terms $(d\sigma_{\nu_\ell e^-}/dT_e)_{\text{SM}}$ and $(d\sigma_{\nu_\ell e^-}/dT_e)_{\text{mag}}$ exhibit quite different dependencies on the experimentally observable electron kinetic energy T_e , as illustrated

in Fig. 9 taken from Balantekin and Vassh (2013) (see also Vogel and Engel (1989); Beda *et al.* (2007)). One can see that small values of the neutrino magnetic moment can be probed by lowering the electron recoil energy threshold. In fact, considering $T_e \ll E_\nu$ in Eq. (4.29) and neglecting the coefficients due to $g_V^{\nu_\ell}$ and $g_A^{\nu_\ell}$ in Eq. (4.26), one can find that $(d\sigma/dT_e)_{\text{mag}}$ exceeds $(d\sigma/dT_e)_{\text{SM}}$ for

$$T_e \lesssim \frac{\pi^2 \alpha^2}{G_F^2 m_e^3} \left(\frac{\mu_{\nu}}{\mu_B} \right)^2. \quad (4.30)$$

D. Effective magnetic moment

In scattering experiments the neutrino is created at some distance from the detector as a flavor neutrino, which is a superposition of massive neutrinos. Therefore, the magnetic moment that is measured in these experiment is not that of a massive neutrino, but it is an effective magnetic moment which takes into account neutrino mixing and the oscillations during the propagation between source and detector (Grimus and Stockinger, 1998; Beacom and Vogel, 1999).

Let us consider an initial neutrino with flavor $\ell = e, \mu, \tau$, which is described by the flavor state in Eq. (2.34). The state of the neutrino which is detected through a scattering process at a space-time distance (\vec{L}, T) from the source is given by the superposition of massive neutrinos in the first line of Eq. (2.38). Considering an incoming left-handed neutrino, the amplitude of ν_j production in low- q^2 electromagnetic scattering of a neutrino which has traveled a space-time distance (\vec{L}, T) from a source of ν_ℓ is

$$\begin{aligned} A_{\ell j}(\vec{L}, T) \propto & \sum_k U_{\ell k}^* e^{-iE_k T + i\vec{p}_k \cdot \vec{L}} \\ & \times \sum_{h_j} u_j^{(h_j)} \sigma_{\mu\nu} q^\nu (\mu_{jk} + i\epsilon_{jk} \gamma_5) u_k^{(-)}. \quad (4.31) \end{aligned}$$

Since for an incoming ultrarelativistic left-handed neutrino the additional γ^5 in the electric dipole term has only the effect of changing a sign (see Eq. (C6)), the amplitude of $\nu_k \rightarrow \nu_j$ transitions is proportional to $\mu_{jk} - i\epsilon_{jk}$, leading to

$$A_{\ell j}(\vec{L}, T) \propto \sum_k U_{\ell k}^* e^{-iE_k T + i\vec{p}_k \cdot \vec{L}} (\mu_{jk} - i\epsilon_{jk}). \quad (4.32)$$

The total cross section of electromagnetic scattering with an electron or a nucleon is given by

$$\sigma_{\nu_\ell e^-}(\vec{L}, T) \propto \sum_j |A_{\ell j}(\vec{L}, T)|^2. \quad (4.33)$$

Taking into account that for ultrarelativistic neutrinos $T = L$, from the approximation in Eq. (2.40) we obtain

that the cross section is proportional to the squared effective magnetic moment

$$\mu_{\nu_\ell}^2(L, E_\nu) = \sum_j \left| \sum_k U_{\ell k}^* e^{-i\Delta m_{kj}^2 L/2E_\nu} (\mu_{jk} - i\epsilon_{jk}) \right|^2. \quad (4.34)$$

In this expression of the effective μ_{ν_ℓ} one can see that in general both the magnetic and electric dipole moments contribute to the elastic scattering. Note also that, as neutrino oscillations discussed in Section II, the effective magnetic moment $\mu_{\nu_\ell}(L, E_\nu)$ depends on the neutrino squared-mass differences, not on the absolute values of neutrino masses.

Considering antineutrinos, the mixing of antineutrinos is obtained from that of neutrinos with the substitution $U \rightarrow U^*$ (see Eqs. (2.34) and (2.42)). From Eq. (3.48) it follows that the electric and magnetic moments of antineutrinos are obtained with the substitutions $\mu_{jk} \rightarrow -\mu_{jk}^*$ and $\epsilon_{jk} \rightarrow -\epsilon_{jk}^*$. Moreover, we must take into account that incoming antineutrinos are right-handed. Hence, for antineutrinos we have

$$\begin{aligned} \bar{A}_{\ell j}(\vec{L}, T) &\propto \sum_k U_{\ell k} e^{-iE_k T + i\vec{p}_k \cdot \vec{L}} \\ &\times \sum_{h_j} \overline{u_j^{(h_j)}} \sigma_{\mu\nu} q^\nu (\mu_{jk}^* + i\epsilon_{jk}^* \gamma_5) u_k^{(+)}. \end{aligned} \quad (4.35)$$

For an incoming ultrarelativistic right-handed neutrino the additional γ_5 in the electric dipole term has no effect (see Eq. (C6)) and we obtain

$$\begin{aligned} \mu_{\nu_\ell}^2(L, E_\nu) &= \sum_j \left| \sum_k U_{\ell k} e^{-i\Delta m_{kj}^2 L/2E_\nu} (\mu_{jk}^* + i\epsilon_{jk}^*) \right|^2 \\ &= \sum_j \left| \sum_k U_{\ell k}^* e^{i\Delta m_{kj}^2 L/2E_\nu} (\mu_{jk} - i\epsilon_{jk}) \right|^2. \end{aligned} \quad (4.36)$$

Therefore, there can be only a phase difference between $\mu_{\nu_\ell}^2(L, E_\nu)$ and $\mu_{\bar{\nu}_\ell}^2(L, E_\nu)$, which is induced by neutrino oscillations.

As discussed in the following Subsection IV.E, the laboratory experiments which are most sensitive to small values of the effective magnetic moment are reactor and accelerator experiments which detect the elastic scattering of flavor neutrinos on electrons at a short distance from the neutrino source. In this case, the value in Eq. (2.73) of the largest squared-mass difference Δm_A^2 in the standard case of three-neutrino mixing is such that $\Delta m_A^2 L/2E_\nu \ll 1$. Therefore, it is possible to approximate all the exponentials in Eqs. (4.34) and (4.36) with unity and obtain the effective short-baseline mag-

netic moment of flavor neutrinos and antineutrinos

$$\begin{aligned} \mu_{\nu_\ell}^2 &\simeq \mu_{\bar{\nu}_\ell}^2 \simeq \sum_j \left| \sum_k U_{\ell k}^* (\mu_{jk} - i\epsilon_{jk}) \right|^2 \\ &= [U(\mu^2 + \epsilon^2)U^\dagger + 2\text{Im}(U\mu\epsilon U^\dagger)]_{\ell\ell}, \end{aligned} \quad (4.37)$$

where we took into account that $\mu = \mu^\dagger$ and $\epsilon = \epsilon^\dagger$. In this approximation the effective magnetic moment is independent from the neutrino energy and from the source-detector distance.

In the following, when we refer to an effective magnetic moment of a flavor neutrino without indication of a source-detector distance L it is implicitly understood that L is small and the effective magnetic moment is given by Eq. (4.37).

It is interesting to note that flavor neutrinos can have effective magnetic moments even if massive neutrinos are Majorana particles. In this case, since massive Majorana neutrinos do not have diagonal magnetic and electric dipole moments, the effective magnetic moments of flavor neutrinos receive contributions only from the transition dipole moments. For example, in the three-generation case, following Eq. (3.71), we can write μ_{jk} and ϵ_{jk} as

$$\mu_{jk} = i \sum_{m=1}^3 \epsilon_{jkm} \tilde{\mu}_m, \quad \epsilon_{jk} = i \sum_{m=1}^3 \epsilon_{jkm} \tilde{\epsilon}_m, \quad (4.38)$$

with real $\tilde{\mu}_m$ and $\tilde{\epsilon}_m$. Thus, we obtain

$$\mu_{\nu_\ell}^2 \simeq \sum_{k=1}^3 (\tilde{\mu}_k^2 + \tilde{\epsilon}_k^2) - \left| \sum_{k=1}^3 U_{\ell k} (\tilde{\mu}_k - i\tilde{\epsilon}_k) \right|^2. \quad (4.39)$$

Another case in which the effective magnetic moment does not depend on the neutrino energy and on the source-detector distance is when the source-detector distance is much larger than all the oscillation lengths $L_{kj} = 4\pi E_\nu / |\Delta m_{kj}^2|$. In this case the interference terms in Eqs. (4.34) and (4.36) are washed out by the finite energy resolution of the detector, leading to

$$\begin{aligned} \mu_{\nu_\ell}^2(\infty) &\simeq \mu_{\bar{\nu}_\ell}^2(\infty) \simeq \sum_k |U_{\ell k}|^2 \sum_j |\mu_{jk} - i\epsilon_{jk}|^2 \\ &= \sum_k |U_{\ell k}|^2 [(\mu^2)_{kk} + (\epsilon^2)_{kk} + 2\text{Im}(\mu\epsilon)_{kk}]. \end{aligned} \quad (4.40)$$

For three-generations of Majorana neutrinos, from Eq. (4.38) we obtain

$$\mu_{\nu_\ell}^2(\infty) \simeq \mu_{\bar{\nu}_\ell}^2(\infty) \simeq \sum_{k=1}^3 (1 - |U_{\ell k}|^2) (\tilde{\mu}_k^2 + \tilde{\epsilon}_k^2). \quad (4.41)$$

So far, in this Subsection we have considered the effects of neutrino mixing and oscillations on the effective magnetic moment for neutrinos propagating in vacuum. In the case of solar neutrinos, which have been used by

the Super-Kamiokande (Liu *et al.*, 2004) and Borexino (Arpesella *et al.*, 2008) experiments to search for neutrino magnetic moment effects, one must take into account the matter effects discussed in Section II.D. The state which describes the neutrinos emerging from the Sun is the following generalization of the state in Eq. (2.70) which takes into account three-neutrino mixing and the squared-mass hierarchy in Eq. (2.74):

$$|\nu_S\rangle = \sum_{k=1}^3 (U_{ek}^M)^* |\nu_k\rangle, \quad (4.42)$$

with

$$U_{e1}^M = \cos \vartheta_{13} \cos \vartheta_M^0, \quad (4.43)$$

$$U_{e2}^M = \cos \vartheta_{13} \sin \vartheta_M^0, \quad (4.44)$$

$$U_{e3}^M = U_{e3} = \sin \vartheta_{13} e^{-i\delta_{13}}, \quad (4.45)$$

where ϑ_M^0 is the effective mixing angle at the point of neutrino production inside the Sun. Following the same reasoning that led to Eq. (4.34), we obtain that the effective magnetic moment measured by an experiment on Earth is

$$\mu_S^2(L, E_\nu) = \sum_j \left| \sum_k (U_{ek}^M)^* e^{-i\Delta m_{kj}^2 L/2E_\nu} (\mu_{jk} - i\epsilon_{jk}) \right|^2, \quad (4.46)$$

where L is the Sun-Earth distance. Since the Sun-Earth distance is much larger than the oscillation lengths, the interference terms in Eqs. (4.46) are washed out by the finite energy resolution of the detector and we obtain the effective magnetic moment

$$\mu_S^2(E_\nu) = \sum_k |U_{ek}^M|^2 \sum_j |\mu_{jk} - i\epsilon_{jk}|^2. \quad (4.47)$$

This expression is similar to that in Eq. (4.40), but takes into account the effective mixing at the point of neutrino production inside the Sun. Note that μ_S depends on the neutrino energy through the dependence of ϑ_M^0 on E_ν (see Eq. (2.63)). As remarked before Eq. (2.72), in practice we have $\vartheta_M^0 \simeq \vartheta_{12}$ for $E_\nu \lesssim 1$ MeV and $\vartheta_M^0 \simeq \pi/2$ for $E_\nu \gtrsim 5$ MeV. Therefore,

$$\mu_S(E_\nu \lesssim 1 \text{ MeV}) \simeq \mu_{\nu_e}(\infty), \quad (4.48)$$

and

$$\begin{aligned} \mu_S^2(E_\nu \gtrsim 5 \text{ MeV}) &\simeq \cos^2 \vartheta_{13} \sum_j |\mu_{j2} - i\epsilon_{j2}|^2 \\ &+ \sin^2 \vartheta_{13} \sum_j |\mu_{j3} - i\epsilon_{j3}|^2. \end{aligned} \quad (4.49)$$

E. Experimental limits

The constraints on the neutrino magnetic moment in direct laboratory experiments have been obtained so far from the lack of any observable distortion of the recoil electron energy spectrum. Experiments of this type have started in the 50's at the Savannah River Laboratory where the $\bar{\nu}_e e^-$ elastic scattering process was studied (Cowan *et al.*, 1954; Cowan and Reines, 1957; Reines *et al.*, 1976) with somewhat controversial results, as discussed by Vogel and Engel (1989). The most significant experimental limits on the effective magnetic moment μ_{ν_e} which have been obtained in reactor $\bar{\nu}_e e^-$ after about 1990 are listed in Tab. III (some details of the different experimental setups are reviewed in Brogini *et al.* (2012))⁷.

The current best limit on μ_{ν_e} has been obtained in 2012 in the GEMMA experiment at the Kalinin Nuclear Power Plant (Russia) with a 1.5 kg highly pure germanium detector exposed at a $\bar{\nu}_e$ flux of $2.7 \times 10^{13} \text{ cm}^{-2} \text{ s}^{-1}$ at a distance of 13.9 m from the core of a 3 GW_{th} commercial water-moderated reactor (Beda *et al.*, 2012). The competitive TEXONO experiment is based at the Kuo-Sheng Reactor Neutrino Laboratory (Taiwan), where a 1.06 kg highly pure germanium detector was exposed to the flux of $\bar{\nu}_e$ at a distance of 28 m from the core of a 2.9 GW_{th} commercial reactor (Wong *et al.*, 2007)⁸.

Searches for effects of neutrino magnetic moments have been performed also in accelerator experiments. The LAMPF bounds on μ_{ν_e} in Tab. III has been obtained with ν_e from μ^+ decay (Allen *et al.*, 1993). The LAMPF and LSND bounds on μ_{ν_μ} in Tab. III has been obtained with ν_μ and $\bar{\nu}_\mu$ from π^+ and μ^+ decay (Allen *et al.*, 1993; Auerbach *et al.*, 2001). The DONUT collaboration (Schwienhorst *et al.*, 2001) investigated $\nu_\tau e^-$ and $\bar{\nu}_\tau e^-$ elastic scattering, finding the limit on μ_{ν_τ} in Tab. III.

Solar neutrino experiments can also search for a neutrino magnetic moment signal by studying the shape of the electron spectrum (Beacom and Vogel, 1999). The

⁷ An attempt to improve the experimental bound on μ_{ν_e} in reactor experiments was undertaken in Wong *et al.* (2010), where it was suggested that in $\bar{\nu}_e$ interactions on an atomic target the atomic electron binding ("atomic-ionization effect") can significantly increase the electromagnetic contribution to the differential cross section with respect to the free-electron approximation. However, as explained in Appendix D, the dipole approximation used to derive the atomic-ionization effect is not valid for the electron antineutrino cross section in reactor neutrino magnetic moment experiments. Instead, the free electron approximation is appropriate for the interpretation of the data of reactor neutrino experiments and the current constraints in Tab. III cannot be improved by considering the atomic electron binding (Voloshin, 2010; Kouzakov and Studenikin, 2011b,a; Kouzakov *et al.*, 2011b,a, 2012; Chen *et al.*, 2013).

⁸ The TEXONO and GEMMA data have been also used by Baranco *et al.* (2012); Healey *et al.* (2013) to constrain neutrino nonstandard interactions.

Method	Experiment	Limit	CL Reference
Reactor $\bar{\nu}_e e^-$	Krasnoyarsk	$\mu_{\nu_e} < 2.4 \times 10^{-10} \mu_B$	90% Vidyakin <i>et al.</i> (1992)
	Rovno	$\mu_{\nu_e} < 1.9 \times 10^{-10} \mu_B$	95% Derbin <i>et al.</i> (1993)
	MUNU	$\mu_{\nu_e} < 0.9 \times 10^{-10} \mu_B$	90% Daraktchieva <i>et al.</i> (2005)
	TEXONO	$\mu_{\nu_e} < 7.4 \times 10^{-11} \mu_B$	90% Wong <i>et al.</i> (2007)
	GEMMA	$\mu_{\nu_e} < 2.9 \times 10^{-11} \mu_B$	90% Beda <i>et al.</i> (2012)
Accelerator $\nu_e e^-$	LAMPF	$\mu_{\nu_e} < 10.8 \times 10^{-10} \mu_B$	90% Allen <i>et al.</i> (1993)
Accelerator $(\nu_\mu, \bar{\nu}_\mu) e^-$	BNL-E734	$\mu_{\nu_\mu} < 8.5 \times 10^{-10} \mu_B$	90% Ahrens <i>et al.</i> (1990)
	LAMPF	$\mu_{\nu_\mu} < 7.4 \times 10^{-10} \mu_B$	90% Allen <i>et al.</i> (1993)
	LSND	$\mu_{\nu_\mu} < 6.8 \times 10^{-10} \mu_B$	90% Auerbach <i>et al.</i> (2001)
Accelerator $(\nu_\tau, \bar{\nu}_\tau) e^-$	DONUT	$\mu_{\nu_\tau} < 3.9 \times 10^{-7} \mu_B$	90% Schwienhorst <i>et al.</i> (2001)
Solar $\nu_e e^-$	Super-Kamiokande	$\mu_S(E_\nu \gtrsim 5 \text{ MeV}) < 1.1 \times 10^{-10} \mu_B$	90% Liu <i>et al.</i> (2004)
	Borexino	$\mu_S(E_\nu \lesssim 1 \text{ MeV}) < 5.4 \times 10^{-11} \mu_B$	90% Arpesella <i>et al.</i> (2008)

TABLE III Experimental limits for different effective neutrino magnetic moments.

effective magnetic moment μ_S in solar $\nu_e e^-$ scattering experiments is given in Eq. (4.47). Table III gives the limits on obtained in the Super-Kamiokande experiment (Liu *et al.*, 2004) for $\mu_S(E_\nu \gtrsim 5 \text{ MeV})$ and that obtained in the Borexino experiment (Arpesella *et al.*, 2008) for $\mu_S(E_\nu \lesssim 1 \text{ MeV})$ (see Eqs. (4.48) and (4.48)).

Information on neutrino magnetic moments has been obtained also with global fits of solar neutrino data (Joshi and Mohanty, 2002; Grimus *et al.*, 2003; Tortola, 2004). Considering Majorana three-neutrino mixing, Tortola (2004) obtained, at 90% CL,

$$\sqrt{|\mu_{12}|^2 + |\mu_{23}|^2 + |\mu_{31}|^2} < 4.0 \times 10^{-10} \mu_B, \quad (4.50)$$

from the analysis of solar and KamLAND, and

$$\sqrt{|\mu_{12}|^2 + |\mu_{23}|^2 + |\mu_{31}|^2} < 1.8 \times 10^{-10} \mu_B, \quad (4.51)$$

adding the Rovno (Derbin *et al.*, 1993), TEXONO (Li, 2003) and MUNU (Daraktchieva *et al.*, 2003) constraints.

F. Theoretical considerations

There is a gap of many orders of magnitude between the present experimental limits on neutrino magnetic moments of the order of $10^{-11} \mu_B$ (discussed in Section IV.E) and the prediction smaller than about $10^{-19} \mu_B$ in Eq. (4.6) of the minimal extension of the Standard Model with right-handed neutrinos. The hope to reach in the near future an experimental sensitivity of this order of magnitude is very weak, taking into account that the experimental sensitivity of reactor $\bar{\nu}_e e$ elastic scattering experiments have improved by only one order of magnitude during a period of about twenty years (see Vogel and Engel (1989), where a sensitivity of the order of $10^{-10} \mu_B$ is discussed). However, the experimental studies of neutrino magnetic moments are stimulated by the hope that new physics beyond the minimally extended Standard Model with right-handed neutrinos might give

much stronger contributions. One of the examples in which it is possible to avoid the neutrino magnetic moment being proportional to a (small) neutrino mass, that would in principle make a neutrino magnetic moment accessible for experimental observations, is realized in the left-right symmetric models considered at the end of Section IV.A.

Other interesting possibilities of obtaining neutrino magnetic moments larger than the prediction in Eq. (4.6) of the minimal extension the Standard Model with right-handed neutrinos have been considered recently. For example, Mohapatra *et al.* (2004) proposed to probe a class of large extra dimensions models with future reactors searches for neutrino magnetic moments and the results obtained in the Minimal Supersymmetric Standard Model with R -parity violating interactions (Gozdz *et al.*, 2006a,b) show that the Majorana transition magnetic moment might be significantly above the scale of Eq. (4.6). The analysis performed by Aboubrahim *et al.* (2014) of the Dirac neutrino magnetic moment within this framework of a Minimal Supersymmetric Standard Model extension with a vectorlike lepton generation shows that a neutrino magnetic moment as large as $(10^{-12} - 10^{-14}) \mu_B$ can be obtained. These values lie within reach of improved laboratory experiments in the future.

Considering the problem of large neutrino magnetic moments, one can write down a generic relation between the size of a neutrino magnetic moment μ_ν and the corresponding neutrino mass m_ν (Voloshin, 1988; Barr *et al.*, 1990; Pal, 1992; Bell *et al.*, 2005, 2006; Bell, 2007). Suppose that a large neutrino magnetic moment is generated by physics beyond a minimal extension of the Standard Model at an energy scale characterized by Λ . For a generic diagram corresponding to this contribution to μ_ν , one can again use the Feynman graph in Fig. 4(b); the circle in this case denotes effects of new physics beyond the Standard Model. The contribution of this diagram

to the magnetic moment is

$$\mu_\nu \sim \frac{eG}{\Lambda}, \quad (4.52)$$

where e is the electric charge and G is a combination of coupling constants and loop factors. The same diagram of Fig. 4(b) but without the photon line gives a new physics contribution to the neutrino mass

$$\delta m_\nu \sim G\Lambda. \quad (4.53)$$

Combining the estimates (4.52) and (4.53), one can get the relation

$$\delta m_\nu \sim \frac{\Lambda^2}{2m_e} \frac{\mu_\nu}{\mu_B} = \frac{\mu_\nu}{10^{-18}\mu_B} \left(\frac{\Lambda}{1 \text{ TeV}} \right)^2 \text{ eV} \quad (4.54)$$

between the one-loop contribution to the neutrino mass and the neutrino magnetic moment.

It follows that, generally, in theoretical models that predict large values for the neutrino magnetic moment, simultaneously large contributions to the neutrino mass arise. Therefore, a particular fine tuning is needed to get a large value for the neutrino magnetic moment while keeping the neutrino mass within experimental bounds. One of the possibilities (Voloshin, 1988) is based on the idea of suppressing the ratio m_ν/μ_ν with a symmetry: if a $SU(2)_\nu$ symmetry is an exact symmetry of the Lagrangian of a model, because of different symmetry properties of the mass and magnetic moment even a massless neutrino can have a nonzero magnetic moment. If, as it happens in a realistic model, the $SU(2)_\nu$ symmetry is broken and if this breaking is small, the ratio m_ν/μ_ν is also small, giving a natural way to obtain a magnetic moment of the order of $\sim 10^{-11}\mu_B$ without contradictions with the neutrino mass experimental constraints. Several possibilities based on the general idea of Voloshin (1988) were considered by Leurer and Marcus (1990); Babu and Mohapatra (1990a); Georgi and Randall (1990); Ecker *et al.* (1989); Chang *et al.* (1991); Barbieri and Mohapatra (1989).

Another idea of neutrino mass suppression without suppression of the neutrino magnetic moment was discussed by Barr *et al.* (1990) within the Zee model (Zee, 1980), which is based on the Standard Model gauge group $SU(2)_L \times U(1)_Y$ and contains at least three Higgs doublets and a charged field which is a singlet of $SU(2)_L$. For this kind of models there is a suppression of the neutrino mass diagram, while the magnetic moment diagram is not suppressed.

It is possible to show with more general and rigorous considerations (Bell *et al.*, 2005, 2006; Bell, 2007) that the Λ^2 dependence in Eq. (4.54) arises from the quadratic divergence in the renormalization of the dimension-four neutrino mass operator. A general and model-independent upper bound on the Dirac neutrino magnetic moment, which can be generated by an effective theory beyond the Standard Model, has been derived (Bell

et al., 2005, 2006; Bell, 2007) from the demand of absence of fine-tuning of effective operator coefficients and from the current experimental information on neutrino masses. A model with Dirac fermions, scalars and gauge bosons that is valid below the scale Λ and respects the Standard Model $SU(2)_L \times U(1)_Y$ symmetry was considered. Integrating out the physics above the scale Λ , the following effective Lagrangian that involves right-handed neutrinos ν_R , lepton isodoublets and the Higgs doublet can be obtained:

$$\mathcal{L}_{eff} = \sum_{n,j} \frac{C_j^n(\mu)}{\Lambda^{n-4}} \mathcal{O}_j^{(n)}(\mu) + \text{H.c.}, \quad (4.55)$$

where μ is the renormalization scale, $n \geq 4$ denotes the operator dimension and j runs over independent operators of a given dimension. For $n = 4$, a neutrino mass arises from the operator $\mathcal{O}_1^{(4)} = \bar{L}\tilde{\Phi}\nu_R$, where $\tilde{\Phi} = i\sigma_2\Phi^*$. In addition, if the scale Λ is not extremely large with respect to the electroweak scale, an important contribution to the neutrino mass can arise also from higher dimension operators. At this point it is important to note that the combination of the $n = 6$ operators appearing in the Lagrangian (4.55) contains the magnetic moment operator $\bar{\nu}\sigma_{\mu\nu}\nu F^{\mu\nu}$ and also generates a contribution δm_ν to the neutrino mass (Bell *et al.*, 2005, 2006; Bell, 2007). Solving the renormalization group equation from the scale Λ to the electroweak scale, one finds that the contributions to the neutrino magnetic moment and to the neutrino mass are connected to each other by

$$|\mu_\nu^D| = \frac{16\sqrt{2}G_F m_e \delta m_\nu \sin^4 \theta_W}{9\alpha^2 |f| \ln(\Lambda/v)} \mu_B, \quad (4.56)$$

where α is the fine structure constant, v is the vacuum expectation value of the Higgs doublet,

$$f = 1 - r - \frac{2}{3} \tan^2 \theta_W - \frac{1}{3} (1 + r) \tan^4 \theta_W, \quad (4.57)$$

and r is a ratio of effective operator coefficients defined at the scale Λ which is of order unity without fine-tuning. If the neutrino magnetic moment is generated by new physics at a scale $\Lambda \sim 1 \text{ TeV}$ and the corresponding contribution to the neutrino mass is $\delta m_\nu \lesssim 1 \text{ eV}$, then the bound $\mu_\nu \lesssim 10^{-14}\mu_B$ can be obtained. This bound is some orders of magnitude stronger than the constraints from reactor and solar neutrino scattering experiments discussed before.

The model-independent limit on a Majorana neutrino transition magnetic moment μ_ν^M was also discussed by Bell *et al.* (2005, 2006); Bell (2007). However, the limit in the Majorana case is much weaker than that in the Dirac case, because for a Majorana neutrino the magnetic moment contribution to the mass is Yukawa suppressed. The limit on μ_ν^M is also weaker than the present experimental limits if μ_ν^M is generated by new physics at the

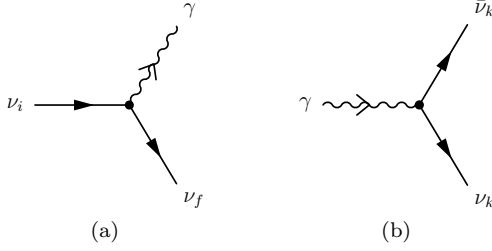


FIG. 10 Feynman diagrams for neutrino radiative decay and Cherenkov radiation (a) and plasmon decay (b).

scale $\Lambda \sim 1$ TeV. An important conclusion of Bell *et al.* (2005, 2006); Bell (2007), based on model-independent considerations of the contributions to μ_ν , is that if a neutrino magnetic moment of order $\mu_\nu \geq 10^{-15} \mu_B$ were observed in an experiment, it would give a proof that neutrinos are Majorana rather than Dirac particles.

V. RADIATIVE DECAY AND RELATED PROCESSES

The magnetic and electric (transition) dipole moments of neutrinos, as well as possible very small electric charges (millicharges), describe direct couplings of neutrinos with photons which induce several observable decay processes. In this Section we discuss the decay processes generated by the diagrams in Fig. 10: the diagram in Fig. 10(a) generates neutrino radiative decay $\nu_i \rightarrow \nu_f + \gamma$ and the processes of neutrino Cherenkov radiation and spin light ($SL\nu$) of a neutrino propagating in a medium; the diagram in Fig. 10(b) generates photon (plasmon) decay to a neutrino-antineutrino pair in a plasma ($\gamma^* \rightarrow \nu\bar{\nu}$).

In Subsections V.A and V.B we review neutrino radiative decay in vacuum and in matter, respectively. In Subsection V.C we discuss neutrino Cherenkov radiation. In Subsection V.D we consider the process of plasmon decay into a neutrino-antineutrino pair, which can be important in dense astrophysical environments as the interior of stars. In Subsection V.E we review the spin light process of a neutrino propagating in a medium.

A. Radiative decay

If the masses of neutrinos are non-degenerate, the radiative decay of a heavier neutrino ν_i into a lighter neutrino ν_f (with $m_i > m_f$) with emission of a photon,

$$\nu_i \rightarrow \nu_f + \gamma, \quad (5.1)$$

may proceed in vacuum (Marciano and Sanda, 1977; Lee and Shrock, 1977; Petcov, 1977; Goldman and Stephenson, 1977; Bilenky and Petcov, 1987; Zatsépin and Smirnov, 1978; Pal and Wolfenstein, 1982). Early discussions of the possible role of neutrino radiative decay in

different astrophysical and cosmological settings can be found in Dicus *et al.* (1977); Sato and Kobayashi (1977); Stecker (1980); Kimble *et al.* (1981); Melott and Sciama (1981); De Rujula and Glashow (1980a).

The neutrino radiative decay process is generated by the interaction in Fig. 5 with a real photon. The decay amplitude is given by

$$\begin{aligned} & \langle \nu_f(p_f, h_f), \gamma(q, \varepsilon) | \int d^4x \mathcal{H}_{\text{em}}^{(\nu)}(x) | \nu_i(p_i, h_i) \rangle \\ &= (2\pi)^4 \delta^4(q - p_i + p_f) \overline{u}^{(h_f)}(p_f) \Lambda_\mu^{fi}(q) u^{(h_i)}(p_i) \varepsilon^\mu, \end{aligned} \quad (5.2)$$

where p_i (p_f) and h_i (h_f) are the four-momentum and helicity of the initial (final) neutrino and q and ε are the four-momentum and polarization four-vectors of the photon. The Dirac δ -function implements energy-momentum conservation.

Taking into account that for a real photon

$$q^2 = 0 \quad \text{and} \quad \varepsilon^\mu q_\mu = 0, \quad (5.3)$$

from the general expression of $\Lambda_\mu(q)$ for Dirac neutrinos in Eq. (3.35) and from Eq. (3.39), we obtain

$$\Lambda_\mu^{fi}(q) \varepsilon^\mu = \mathfrak{q}_{fi} \not{\varepsilon} - i \sigma_{\mu\nu} \varepsilon^\mu q^\nu (\mathfrak{p}_{fi} + i \mathfrak{e}_{fi} \gamma_5), \quad (5.4)$$

where $\mathfrak{q}_{fi} \neq 0$ only if neutrinos are millicharged particles (see Section VII.A). Therefore, the radiative decay of a neutrino ν_i into a lighter neutrino ν_f depends on the corresponding transition charge, magnetic moment and electric moment. Assuming $\mathfrak{q}_{fi} = 0$, the decay rate in the rest frame (rf) of the decaying neutrino ν_i is given by (see Raffelt (1996, 1999a,b))

$$\Gamma_{\nu_i \rightarrow \nu_f + \gamma}^{\text{rf}} = \frac{1}{8\pi} \left(\frac{m_i^2 - m_f^2}{m_i} \right)^3 (|\mathfrak{p}_{fi}|^2 + |\mathfrak{e}_{fi}|^2). \quad (5.5)$$

This expression is valid for both Dirac and Majorana neutrinos, because both can have transition magnetic and electric moments and the corresponding expression (3.62) for $\Lambda_\mu(q)$ in the Majorana case is equivalent to that in Eq. (3.35) for Dirac neutrinos.

The transition magnetic and electric dipole moments of Dirac neutrinos in the minimal extension of the Standard Model with right-handed neutrinos are given approximately by Eq. (4.8). In this case, the radiative decay rate is given by (Marciano and Sanda, 1977; Lee and Shrock, 1977; Petcov, 1977; Goldman and Stephenson, 1977; Bilenky and Petcov, 1987; Zatsépin and Smirnov, 1978; Pal and Wolfenstein, 1982)

$$\begin{aligned} \Gamma_{\nu_i^D \rightarrow \nu_f^D + \gamma}^{\text{rf}} &\simeq \frac{\alpha G_F^2}{128\pi^4} \left(\frac{m_i^2 - m_f^2}{m_i} \right)^3 (m_i^2 + m_f^2) \\ &\times \left| \sum_{l=e,\mu,\tau} U_{li}^* U_{lf} \frac{m_l^2}{m_W^2} \right|^2. \end{aligned} \quad (5.6)$$

Hence, the radiative decay rate is suppressed by the small phase space due to the smallness of neutrino masses, by the proportionality of the magnetic (electric) transition moment to the sum (difference) of the masses of the two neutrinos involved in the decay and by a coefficient which smaller than $(m_\tau/m_W)^4 \simeq 2 \times 10^{-7}$. Note, however, that there are models (see, for instance, Petcov (1982)) in which the neutrino radiative decay rate (as well as the magnetic moment discussed above) of a non-standard Dirac neutrino are much larger than those predicted in the minimally extended Standard Model.

The expression of the decay rate for Majorana neutrinos in the simplest extensions of the Standard Model (without taking into account model-dependent contributions of the scalar sector) can be easily derived from the expressions in Eqs. (4.15) and (4.16) of the Majorana magnetic and electric transition moments. In the case of CP conservation, from Eqs. (4.20) and (4.21) it follows that the decay process is induced purely by the neutrino electric or magnetic transition dipole moment if the CP phases of ν_i and ν_f are, respectively, equal or opposite.

For numerical estimations it is convenient to express the lifetime $\tau_{\nu_i \rightarrow \nu_f + \gamma} = \Gamma_{\nu_i \rightarrow \nu_f + \gamma}^{-1}$ in the following form:

$$\tau_{\nu_i \rightarrow \nu_f + \gamma}^{\text{rf}} \simeq 0.19 \left(\frac{m_i^2}{m_i^2 - m_f^2} \right)^3 \left(\frac{\text{eV}}{m_i} \right)^3 \left(\frac{\mu_B}{\mu_{fi}^{\text{eff}}} \right)^2 \text{ s}, \quad (5.7)$$

with the effective neutrino magnetic moment

$$\mu_{fi}^{\text{eff}} = \sqrt{|\mu_{fi}|^2 + |\epsilon_{fi}|^2}. \quad (5.8)$$

Since μ_{fi}^{eff} is very small, the lifetime in Eq. 5.7 is very long. Indeed, in the case of Dirac neutrinos in the minimal extension of the Standard Model with right-handed neutrinos, considering only the dominant τ contribution in Eq. (5.6) and neglecting m_f , we obtain

$$\tau_{\nu_i^D \rightarrow \nu_f^D + \gamma}^{\text{rf}} \simeq \frac{3.5 \times 10^{43} \text{ s}}{|U_{\tau i}|^2 |U_{\tau f}|^2} \left(\frac{\text{eV}}{m_i} \right)^5. \quad (5.9)$$

This lifetime is much larger than the age of the Universe, which is about $4.3 \times 10^{17} \text{ s}$ (Beringer *et al.*, 2012).

The neutrino radiative decay can be constrained by the absence of decay photons in reactor $\bar{\nu}_e$ and solar ν_e fluxes. The limits on μ_{fi}^{eff} that are obtained from these considerations are much weaker than those obtained from neutrino scattering terrestrial experiments. Stronger constraints on μ_{fi}^{eff} (though still weaker than those obtained in terrestrial experiments) are obtained from the neutrino decay limit set by SN 1987A (Kolb and Turner, 1989; Jaffe and Turner, 1997) and from the measurements of the diffuse cosmic infrared background and those of the cosmic microwave background (Cowsik, 1977; Sato and Kobayashi, 1977; Dicus *et al.*, 1978; De Rujula and Glashow, 1980b; Stecker, 1980; Kimble *et al.*, 1981; Dolgov and Zeldovich,

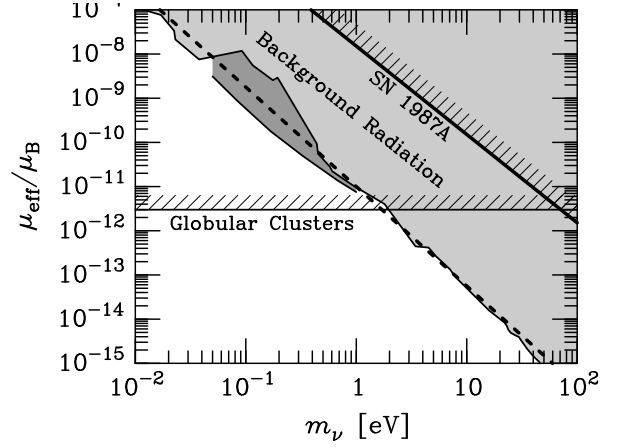


FIG. 11 Astrophysical limits on neutrino transition moments (Raffelt, 1996, 1999a,b).

1981; Ressel and Turner, 1990; Biller *et al.*, 1998; Raffelt, 1998; Masso and Toldra, 1999; Mirizzi *et al.*, 2007). These limits, shown in Fig. 11, can be expressed as (see Raffelt (1996, 1999a,b))

$$\frac{\mu_{fi}^{\text{eff}}}{\mu_B} < \begin{cases} 0.9 \times 10^{-1} \left(\frac{\text{eV}}{m_\nu} \right)^2 & \text{Reactor } (\bar{\nu}_e), \\ 0.5 \times 10^{-5} \left(\frac{\text{eV}}{m_\nu} \right)^2 & \text{Sun } (\nu_e), \\ 1.5 \times 10^{-8} \left(\frac{\text{eV}}{m_\nu} \right)^2 & \text{SN 1987A (all flavors),} \\ 1.0 \times 10^{-11} \left(\frac{\text{eV}}{m_\nu} \right)^{9/4} & \text{Cosmic background} \\ & \text{(all flavors).} \end{cases} \quad (5.10)$$

Let us finally recall several studies of the effect of neutrino radiative decay on primordial Big-Bang Nucleosynthesis (Sato and Kobayashi, 1977; Dicus *et al.*, 1978; Miyama and Sato, 1978; Audouze *et al.*, 1985; Terasawa *et al.*, 1988) (see Dolgov (2002)).

B. Radiative decay in matter

As explained in Subsection II.D, the evolution of neutrinos propagating in matter is affected by the potential generated by the coherent forward elastic scattering with the particles in the medium. It turns out that the coherent interaction with an electron background induces the radiative decay in Eq. (5.1) with a rate that is not suppressed by the GIM mechanism as the decay rate in vacuum in Eq. (5.6) (D'Olivo *et al.*, 1990). Following the approach of Giunti *et al.* (1992), the process of radiative decay in an electron background can be represented by the two Feynman diagrams in Fig. 12 which are obtained from the CC potential diagram in Fig. 1(a) attaching a final photon line at the initial or final electron line. As in the case of the calculation of the potential (see Giunti and Kim (2007)), the coherent contribution of the electron background is obtained by considering equal initial

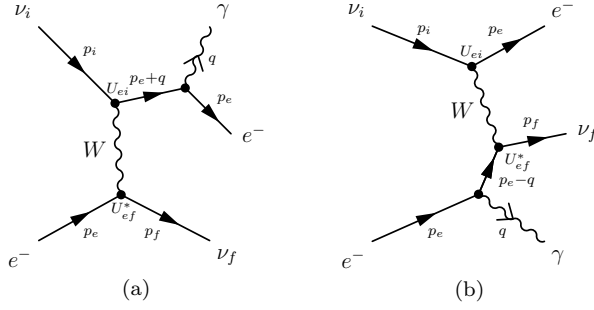


FIG. 12 Feynman diagrams of the coherent contribution of background electrons to the radiative decay $\nu_i(p_i) \rightarrow \nu_f(p_f) + \gamma(q)$ in matter.

and final four-momenta of the electron. The resulting decay rate in the rest frame of the electron background is

$$\Gamma_{\nu_i \rightarrow \nu_f + \gamma}^{\text{mat}} = \frac{\alpha G_F^2 N_e^2}{2m_e^2} \left(\frac{m_i^2 - m_f^2}{m_i} \right) |U_{ei}|^2 |U_{ef}|^2 F(v_i), \quad (5.11)$$

where N_e is the electron number density, $v_i = |\vec{p}_i|/E_i$ is the velocity of the initial neutrino, and

$$F(v_i) = \sqrt{1 - v_i^2} \left[\frac{2}{v_i} \ln \left(\frac{1 + v_i}{1 - v_i} \right) - 3 + \frac{m_f^2}{m_i^2} \right]. \quad (5.12)$$

In the realistic case of ultrarelativistic initial neutrinos, we have

$$F(v_i) \xrightarrow{v_i \rightarrow 1} 4 m_i / E_i. \quad (5.13)$$

Note that the matter-induced radiative decay is independent from the Dirac or Majorana nature of neutrinos, because it is generated by the coherent weak interactions with matter, which are the same for left-handed neutrinos.

Neglecting the final neutrino mass in Eq. (5.11), the numerical value of the lifetime $\tau_{\nu_i \rightarrow \nu_f + \gamma}^{\text{mat}} = (\Gamma_{\nu_i \rightarrow \nu_f + \gamma}^{\text{mat}})^{-1}$ for ultrarelativistic initial neutrinos is given by

$$\tau_{\nu_i \rightarrow \nu_f + \gamma}^{\text{mat}} \simeq \frac{4.0 \times 10^{30} \text{ s}}{|U_{ei}|^2 |U_{ef}|^2} \left(\frac{\text{eV}}{m_i} \right)^2 \left(\frac{E_i}{\text{MeV}} \right) \left(\frac{N_A \text{ cm}^{-3}}{N_e} \right)^2. \quad (5.14)$$

In order to compare the radiative lifetime in matter in Eq. (5.14) with the radiative lifetime in vacuum in Eq. (5.9), obtained in the minimal extension of the Standard Model with right-handed neutrinos, we must take into account the Lorentz boost factor $\gamma_i = E_i/m_i$ from the rest frame of the decaying neutrino to the rest frame

of the electron background:

$$\frac{\tau_{\nu_i \rightarrow \nu_f + \gamma}^{\text{mat}}}{\gamma_i \tau_{\nu_i^D \rightarrow \nu_f^D + \gamma}^{\text{rf}}} \simeq 1.1 \times 10^{-19} \left(\frac{|U_{\tau i}|^2 |U_{\tau f}|^2}{|U_{ei}|^2 |U_{ef}|^2} \right) \times \left(\frac{m_i}{\text{eV}} \right)^4 \left(\frac{N_A \text{ cm}^{-3}}{N_e} \right)^2. \quad (5.15)$$

Therefore, the radiative decay rate in an electron background is many orders of magnitude larger than the radiative decay rate in vacuum in the minimal extension of the Standard Model with right-handed neutrinos. However, the large value of the lifetime in Eq. (5.14) indicate that it is very difficult, if not impossible, to find a realistic application of this effect.

So far we have considered the radiative decay rate in a background of electrons, assuming that the temperature is not very high. For a temperature $T \gg m_e$ both electrons and positrons are present in the background and the radiative decay rate is given by D'Olivo *et al.* (1990)

$$\Gamma_{\nu_i \rightarrow \nu_f + \gamma}^{(T \gg m_e)} = \frac{\alpha G_F^2 T^4}{72} \left(\frac{m_i^2 - m_f^2}{m_i} \right) |U_{ei}|^2 |U_{ef}|^2 F(v_i). \quad (5.16)$$

Neglecting the final neutrino mass, for ultrarelativistic initial neutrinos we have

$$\tau_{\nu_i \rightarrow \nu_f + \gamma}^{(T \gg m_e)} \simeq \frac{1.2 \times 10^{16} \text{ s}}{|U_{ei}|^2 |U_{ef}|^2} \left(\frac{\text{eV}}{m_i} \right)^2 \left(\frac{E_i}{\text{MeV}} \right) \left(\frac{\text{MeV}}{T} \right)^4. \quad (5.17)$$

Therefore, in this case the radiative decay in matter is enormously faster than that in vacuum in the minimal extension of the Standard Model with right-handed neutrinos:

$$\frac{\tau_{\nu_i \rightarrow \nu_f + \gamma}^{(T \gg m_e)}}{\gamma_i \tau_{\nu_i^D \rightarrow \nu_f^D + \gamma}^{\text{rf}}} \simeq 3.3 \times 10^{-34} \left(\frac{|U_{\tau i}|^2 |U_{\tau f}|^2}{|U_{ei}|^2 |U_{ef}|^2} \right) \times \left(\frac{m_i}{\text{eV}} \right)^4 \left(\frac{\text{MeV}}{T} \right)^4. \quad (5.18)$$

Let us finally mention that: Nieves and Pal (1997) calculated the radiative decay rate of neutrinos propagating in a thermal background of electrons and photons, taking into account the effect of the stimulated emission of photons in the thermal bath; Grasso and Semikoz (1999) calculated the decay rate of a neutrino induced by the emission or absorption of a photon in a plasma taking into account the effective mass of the photons (plasmons); Skobelev (1995), Zhukovsky *et al.* (1996) and Kachelriess and Wunner (1997) calculated the radiative decay rate of neutrinos propagating in magnetic fields; Ternov and Eminov (2003, 2013) calculated the radiative decay rate of neutrinos propagating in a magnetized plasma.

C. Cherenkov radiation

It is well known that a charged particle moving through medium at a velocity greater than the speed of light in medium, $v > c/n$ (n is the medium refractive index), can emit the Cherenkov radiation. In much the same way neutrinos with an anomalous magnetic moment (and/or an electric dipole moment) propagating in a medium with a velocity larger than the phase velocity of light can also produce the electromagnetic radiation by the Cherenkov mechanism. This possibility was first discussed in Radomski (1975) to advance the solution of the solar neutrino problem by lowering the expected counting rate because neutrinos might lose reasonable amount of energy in solar matter by the Cherenkov mechanism. However, the calculated magnetic-moment Cherenkov process was found to be too small to reduce significantly the solar neutrino flux.

The amplitude of the neutrino magnetic moment Cherenkov radiation process

$$\nu_L(p) \rightarrow \nu_R(p') + \gamma(k) \quad (5.19)$$

is given by

$$M = \frac{\mu}{n} \bar{u}(p', s') \sigma_{\mu\nu} k^\mu u(p, s) \epsilon^\nu(k, \lambda), \quad (5.20)$$

where $p = (p_0, \vec{p})$ and $p' = (p'_0, \vec{p}')$ are the four-momenta of initial and final neutrinos and $k = (\omega, \vec{k})$ and ϵ are the four-momenta and polarization four-vector of the emitted photon. The transition rate of the process derived in Grimus and Neufeld (1993); Mohanty and Samal (1996) is given by

$$\Gamma = \frac{1}{2(2\pi)^2 p_0} \int |M|^2 \frac{d^3 p'}{2p'_0} \frac{d^3 k}{2\omega} \delta^{(4)}(p - p' - k). \quad (5.21)$$

After integration with use of the δ function the rate is obtained in the form

$$\Gamma = \frac{1}{16\pi} \int |M|^2 \frac{k^2 dk \sin \theta d\theta}{n\omega^2 p_0^2} \delta\left(\cos \theta - \frac{2\omega p_0 - k^2}{2|\vec{p}||\vec{k}|}\right), \quad (5.22)$$

where θ is the angle between the emitted photon and the direction of the initial neutrino propagation. The δ function constraints the photon emission angle to have the value

$$\cos \theta = \frac{1}{nv} \left(1 + (n^2 - 1) \frac{\omega}{2p_0}\right), \quad (5.23)$$

where $v = |\vec{p}|/p_0$ is the neutrino velocity and $k^2 = -(n^2 - 1)\omega^2$. Obviously, the kinematically allowed directions of the Cherenkov radiation are for $|\cos \theta| \leq 1$. After performing integrals and accounting for (5.23) the

following expression can be obtained for the Cherenkov process rate

$$\Gamma = \frac{\mu^2}{4\pi p_0^2} \int_{\omega_1}^{\omega_2} \left\{ \left[\frac{(n^2 - 1)^2}{n^2} p_0^2 + (n^2 - 1) m_\nu^2 \right] \omega^2 - \frac{(n^2 - 1)^2}{n^2} p_0 \omega^3 - \frac{(n^2 - 1)^3}{n^2} \omega^4 \right\} d\omega, \quad (5.24)$$

where the integral limits ω_1 and ω_2 determine the range of the allowed Cherenkov radiation frequencies.

The obtained general expression (5.24) can be used for analysis of possible phenomenological consequences of the Cherenkov radiation due to the neutrino magnetic moment in different environments. In particular, as it follows from estimations performed in Grimus and Neufeld (1993) it is expected that if the solar neutrinos with an effective magnetic moment of $\mu \sim 3 \times 10^{-11}$ arrive to a 1km^3 water detector at the earth then there will be around 5 photons per day in the range of visible light produced by the neutrino magnetic moment Cherenkov mechanism.

Of course, the considered Cherenkov mechanism is of interest for other astrophysical applications. The discussed process flips the neutrino helicity via which active left-handed neutrinos are converted to sterile right-handed neutrinos. This can have important consequences, for instance, for a supernova core. In Mohanty and Samal (1996) it is assumed that the luminosity of such sterile neutrinos is less than the total energy $10^{53}\text{ergs sec}^{-1}$ and an upper bound on the effective neutrino moment on the level of $0.3 \times 10^{-13} \mu_B$ is obtained.

A similar to what has been discussed above Cherenkov process can be opened to a neutrino with non-zero effective magnetic moment in a magnetic field. The presence of a background magnetic field alone (without presence of matter) modifies the photon dispersion relation and so the neutrino magnetic moment Cherenkov process becomes feasible. This possibility was first discussed in Galtsov and Nikitina (1972); Skobelev (1976). In a more recent paper (Ioannisian and Raffelt, 1997) a detailed analysis of the Cherenkov radiation by a neutrino magnetic moment in a magnetic field as well as some related references are presented. The obtained results are applied to the strongest magnetic fields known in nature as that near pulsars. However, even if the magnetic field is as strong as the critical field $B_{\text{cr}} = m_e^2/e = 4.41 \times 10^{13}\text{G}$ due to its spatial extensions near pulsars only for tens of kilometers most of escaping from the pulsar neutrinos have no chance to emit photons by the Cherenkov process.

There is also another possible mechanism of the electromagnetic radiation also termed the Cherenkov radiation by a neutrino in medium discussed in Sawyer (1992); D'Olivo *et al.* (1996). This mechanism is based on the point first realized in Oraevsky *et al.* (1986) that neutrinos moving in media acquire an electric charge as

a consequence of their weak interaction with particles of the background. Thus, due to this interaction with the background a neutrino becomes a charged particle and can emit photons due to the Cherenkov mechanism. Note that this effect exist even for massless neutrinos and no physics beyond the Standard Model is needed. For the range of optical photons, the magnetic moment Cherenkov radiation considered in Grimus and Neufeld (1993) is much larger than the Cherenkov radiation due to the induced charge. However, if to consider photons with higher energies from the allowed kinematic range then the effect of the Cherenkov radiation due to the induced neutrino charge becomes important. This might be of interest for applications in astrophysics.

For completeness, we would like to mention that other processes characterized by the same signature of Eq. (5.1) have been considered previously (for a review of the literature see Ioannisian and Raffelt (1997); Lobanov and Studenikin (2003); Studenikin (2004, 2007, 2006b)):

- i) The photon radiation by a massless neutrino ($\nu_i \rightarrow \nu_f + \gamma$, with $i = f$) due to the vacuum polarization loop diagram in the presence of an external magnetic field (Galtsov and Nikitina, 1972).
- ii) The photon radiation by a massive neutrino with nonvanishing magnetic moment in constant magnetic and electromagnetic wave fields (Borisov *et al.*, 1988, 1989b; Skobelev, 1976, 1991).
- iii) The Cherenkov radiation due to the nonvanishing neutrino magnetic moment in an homogeneous and infinitely extended medium, which is only possible if the speed of the neutrino is larger than the speed of light in the medium (Radomski, 1975; Grimus and Neufeld, 1993).
- iv) The transition radiation due to a nonvanishing neutrino magnetic moment which would be produced when the neutrino crosses the interface of two media with different refractive indices (Sakuda, 1994; Sakuda and Kurihara, 1995; Grimus and Neufeld, 1995).
- v) The Cherenkov radiation of a massless neutrino due to its induced charge in a medium (Oraevsky *et al.*, 1986; D'Olivo *et al.*, 1996)⁹.
- vi) The Cherenkov radiation of massive and massless neutrinos in a magnetized medium (Mohanty and Samal, 1996; Ioannisian and Raffelt, 1997).
- vii) The neutrino radiative decay ($\nu_i \rightarrow \nu_f + \gamma$, with $i \neq f$) in external fields and media (see Giunti *et al.* (1991); Gvozdev *et al.* (1992); Skobelev (1995); Zhukovsky *et al.* (1996); Kachelriess and Wunner (1997); Ternov and Eminov (2003)).

D. Plasmon decay into a neutrino-antineutrino pair

The most interesting process, for the purpose of constraining neutrino electromagnetic properties, is the photon (plasmon) decay into a neutrino-antineutrino pair, $\gamma^* \rightarrow \nu + \bar{\nu}$ (Bernstein *et al.*, 1963; Sutherland *et al.*, 1976). This plasmon process becomes kinematically allowed in media, because a photon with the dispersion relation $\omega_\gamma^2 - \vec{k}_\gamma^2 > 0$ roughly behaves as a particle with an effective mass. For example, photons in a nonrelativistic plasma have the dispersion relation $\omega_\gamma^2 - \vec{k}_\gamma^2 = \omega_P^2$, where $\omega_P = 4\pi\alpha N_e/m_e$ is the plasma frequency (Raffelt, 1996). For $\omega_P > 2m_\nu$ the plasmon decay $\gamma^* \rightarrow \nu + \bar{\nu}$ is kinematically possible.

The plasmon decay rate is (Sutherland *et al.*, 1976; Raffelt, 1996)

$$\Gamma_{\gamma^* \rightarrow \nu \bar{\nu}} = \frac{\mu_\nu^2}{24\pi} Z \frac{(\omega_\gamma^2 - k_\gamma^2)^2}{\omega_\gamma}, \quad (5.25)$$

where μ_ν is the effective magnetic moment

$$\mu_\nu = \sum_{k,j} (|\mu_{kj}|^2 + |\epsilon_{kj}|^2). \quad (5.26)$$

The quantity Z is a renormalization factor which depends on the polarization of the plasmon. For transverse plasmons $\omega_\gamma^2 - k_\gamma^2 = \omega_P^2$ and $Z = 1$, whereas for longitudinal plasmons $\omega_\gamma \simeq \omega_P$ and $Z \simeq (1 - k_\gamma^2/\omega_P^2)^{-1}$ (Raffelt, 1996).

The process of plasmon decay into a neutrino-antineutrino pair was first considered by Bernstein *et al.* (1963) as a new energy-loss channel for the Sun. In general, a plasmon decay in a star liberates the energy ω_γ in the form of neutrinos that freely escape the stellar environment. The corresponding energy-loss rate per unit volume is

$$Q_{\gamma^* \rightarrow \nu \bar{\nu}} = \frac{g}{(2\pi)^3} \int \omega_\gamma f_{k_\gamma} \Gamma_{\gamma \rightarrow \nu \bar{\nu}} d^3 k_\gamma, \quad (5.27)$$

where f_{k_γ} is the photon Bose-Einstein distribution function and $g = 2$ is the number of polarization states.

The requirement that the plasmon-decay energy-loss channel does not exceed the standard solar model luminosity leads to the constraint (Raffelt, 1996, 1999a,b)

$$\mu_\nu \lesssim 4 \times 10^{-10} \mu_B. \quad (5.28)$$

⁹ Note that the neutrino electromagnetic properties are in general affected by the external environment. In particular, a neutrino can acquire an electric charge in magnetized matter (Oraevsky *et al.*, 1986; D'Olivo *et al.*, 1996) and the neutrino magnetic moment depends on the strength of external electromagnetic fields (Borisov *et al.*, 1985, 1989b; Masood *et al.*, 2002; Egorov *et al.*, 1999). A recent study of the neutrino electromagnetic vertex in magnetized matter can be found in Nieves (2003). See also Studenikin (2004, 2007) for a review of neutrino interactions in external electromagnetic fields.

However, the tightest astrophysical bound on μ_ν comes from the constraints on the possible delay of helium ignition of red giant star in globular clusters due to the cooling induced the plasmon-decay energy loss. From the lack of observational evidence of this effect, the following limit has been found (Raffelt, 1990c,a; Raffelt and Weiss, 1992):

$$\mu_\nu \lesssim 3 \times 10^{-12} \mu_B. \quad (5.29)$$

See also Castellani and Degl’Innocenti (1993); Catelan *et al.* (1995). Recently the limit has been updated by Viaux *et al.* (2013) using state-of-the-art astronomical observations and stellar evolution codes, with the results

$$\mu_\nu < \begin{cases} 2.6 \times 10^{-12} \mu_B & (68\% \text{ CL}), \\ 4.5 \times 10^{-12} \mu_B & (95\% \text{ CL}). \end{cases} \quad (5.30)$$

This astrophysical constraint on a neutrino magnetic moment is applicable to both Dirac and Majorana neutrinos and constraints all diagonal and transition dipole moments according to Eq. (5.26).

It has also been shown by (Heger *et al.*, 2009) that the additional cooling due to processes induced by neutrino magnetic moments (plasmon decay $\gamma^* \rightarrow \nu\bar{\nu}$, photo processes $\gamma e^- \rightarrow e^- \nu\bar{\nu}$, pair processes $e^+e^- \rightarrow \nu\bar{\nu}$, bremsstrahlung $e^-(Ze) \rightarrow (Ze)e^- \nu\bar{\nu}$) generate qualitative changes to the structure and evolution of stars with masses between 7 and 18 solar masses, rather than simply changing the time scales of their burning. The resulting sensitivity to the neutrino magnetic moment has been estimated by Heger *et al.* (2009) to be at the level of $(2-4) \times 10^{-11} \mu_B$.

E. Spin light

It is known from electrodynamics that a system with zero electric charge but non-zero magnetic moment can produce the electromagnetic radiation. This phenomenon in classical theory is due to the magnetic moment rotation and it is termed the magnetic dipole radiation. Its power is proportional to the second derivative of the magnetic moment of an emitting system squared.

The similar mechanism of radiation exist for a neutrino. It was shown (Lobanov and Studenikin, 2003) that a neutrino with nonzero magnetic moment in presence of matter emit electromagnetic radiation termed the spin light of neutrino, $SL\nu$. In general, the $SL\nu$ is an electromagnetic radiation that can be emitted by a neutrino due to the neutrino magnetic (or electric) moments, both diagonal and transition, when the particle moves in the background matter. Within a quasi-classical treatment, the $SL\nu$ was first proposed and studied (Lobanov and Studenikin, 2003, 2004; Grigoriev *et al.*, 2005a) on the basis of the developed Lorentz invariant approach to the neutrino spin evolution that implies the use of the generalized Bargmann-Michel-Telegdi equation (Egorov *et al.*,

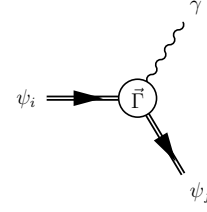


FIG. 13 The spin light of neutrino ($SL\nu$) radiation diagram.

2000; Lobanov and Studenikin, 2001) (for further details see Appendix F).

It should be mentioned that the $SL\nu$ in matter is different from the neutrino Cherenkov radiation in matter mentioned above, because it can exist even when the emitted photon refractive index is equal to unity. The $SL\nu$ radiation is due to radiation of the neutrino by its own, rather than radiation of the background particles.

As it was clear from the very beginning (Lobanov and Studenikin, 2003), the $SL\nu$ is a quantum phenomenon by its nature. The quantum theory of this radiation has been elaborated (Studenikin and Ternov, 2005; Grigoriev *et al.*, 2005; Grigoriev *et al.*, 2005b, 2006b; Studenikin, 2007; Grigoriev *et al.*, 2012b) (see also Lobanov (2005b,a)) within development (Studenikin, 2006b,a, 2008; Grigoriev *et al.*, 2009) of a quite powerful method that implies the use of the exact solutions of the modified Dirac equation for the neutrino wave function in matter in evaluations of rates and other characteristics of different processes in the background matter.

The corresponding Feynman diagram of the $SL\nu$ processes is shown in Fig. 13, where the neutrino initial (ψ_i) and final (ψ_f) states (indicated by “broad lines”) are exact solutions of the corresponding Dirac equations accounting exactly for the interaction with matter. The neutrino wave functions and energy spectrum are given by Eqs.(G16) and (G17) of Appendix F. Here we consider a generic flavor neutrino with an effective magnetic moment μ_ν and effective mass m_ν . Within the quantum treatment the $SL\nu$ process for the relativistic neutrino should be considered as the transition process from more energetic neutrino initial state to low lying final state with emission of a photon (Studenikin and Ternov, 2005; Grigoriev *et al.*, 2005). This process proceeds with the neutrino helicity flip.

The amplitude of the $SL\nu$ process is given by (Studenikin and Ternov, 2005)

$$S_{fi} = -\mu_\nu \sqrt{4\pi} \int d^4x \bar{\psi}_f(x) (\vec{\Gamma} \cdot \vec{\varepsilon}^*) \frac{e^{ikx}}{\sqrt{2\omega L^3}} \psi_i(x), \quad (5.31)$$

where L^3 is the normalization volume,

$$\vec{\Gamma} = i\omega \{ [\vec{\Sigma} \times \vec{\mathcal{A}}] + i\gamma^5 \vec{\Sigma} \}, \quad \text{with} \quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \quad (5.32)$$

and $k^\mu = (\omega, \vec{k})$ and $\vec{\varepsilon}$ are the photon momentum and polarization vectors, $\vec{\varkappa} = \vec{k}/\omega$ is the unit vector pointing in the direction of propagation of the emitted photon. Here $\psi_i(x)$ and $\psi_f(x)$ are the initial and final neutrino wave functions in presence of matter obtained as the exact solutions of the effective Dirac equation

$$\left\{ i\gamma_\mu \partial^\mu - \frac{1}{2}\gamma_\mu(1 + \gamma_5)\tilde{f}^\mu - m_\nu \right\} \psi_{i,f}(x) = 0, \quad (5.33)$$

(see Eqs. (G16) and (G17) of Appendix G). From the energy-momentum conservation

$$p_0 = p'_0 + \omega, \quad \vec{p} = \vec{p}' + \vec{\varkappa}, \quad (5.34)$$

where p_0, \vec{p} and p'_0, \vec{p}' are the initial and final neutrino energy and momenta, it follows that the photon energy,

$$\omega = \frac{2\tilde{N}m_\nu p \left[(p_0 - \tilde{N}m_\nu) - (p + \tilde{N}m_\nu) \cos \theta \right]}{(p_0 - \tilde{N}m_\nu - p \cos \theta)^2 - (\tilde{N}m_\nu)^2}, \quad (5.35)$$

where $p = |\vec{p}|$, is rather complicated function of the initial neutrino energy and momentum, the angle θ between $\vec{\varkappa}$ and the initial neutrino propagation and the matter density parameter \tilde{N} that in case of the electron neutrino moving in matter composed of electrons, protons and neutrons is

$$\tilde{N} = \frac{G_F}{2\sqrt{2}} \left[N_e(1 + 4\sin^2 \theta_W) + N_p(1 - 4\sin^2 \theta_W) - N_n \right], \quad (5.36)$$

where N_e, N_p and N_n are the number densities of the background electrons, protons and neutrons, respectively. From the amplitude (5.31) and the photon energy (5.35) the $SL\nu$ transition rate and total radiation power can be obtained:

$$\Gamma = \mu_\nu^2 \int_0^\pi \frac{\omega^3}{1 + \tilde{\beta}'y} S \sin \theta d\theta, \quad (5.37)$$

$$I = \mu_\nu^2 \int_0^\pi \frac{\omega^4}{1 + \tilde{\beta}'y} S \sin \theta d\theta, \quad (5.38)$$

where

$$S = (\tilde{\beta}\tilde{\beta}' + 1)(1 - y \cos \theta) - (\tilde{\beta} + \tilde{\beta}')(\cos \theta - y), \quad (5.39)$$

and

$$\tilde{\beta} = \frac{p + \tilde{N}m_\nu}{p_0 - \tilde{N}m_\nu}, \quad \tilde{\beta}' = \frac{p' - \tilde{N}m_\nu}{p'_0 - \tilde{N}m_\nu}, \quad y = \frac{\omega - p \cos \theta}{p'}, \quad (5.40)$$

where $p' = |\vec{p}'|$. For the case of a relativistic neutrino, $p \gg m$, the total rate and power for different limiting values of momentum $p \gg m_\nu$ and matter density \tilde{N} are

$$\Gamma = \begin{cases} \frac{64}{3} \mu_\nu^2 \tilde{N}^3 p^2 m_\nu & \text{for } \tilde{N} \ll \frac{m_\nu}{p}, \\ 4 \mu_\nu^2 \tilde{N}^2 m_\nu^2 p & \text{for } \frac{m_\nu}{p} \ll \tilde{N} \ll \frac{p}{m_\nu}, \\ 4 \mu_\nu^2 \tilde{N}^3 m_\nu^3 & \text{for } \tilde{N} \gg \frac{p}{m_\nu}, \end{cases} \quad (5.41)$$

$$I = \begin{cases} \frac{128}{3} \mu_\nu^2 \tilde{N}^4 p^4 & \text{for } \tilde{N} \ll \frac{m_\nu}{p}, \\ \frac{4}{3} \mu_\nu^2 \tilde{N}^2 m_\nu^2 p^2 & \text{for } \frac{m_\nu}{p} \ll \tilde{N} \ll \frac{p}{m_\nu}, \\ 4 \mu_\nu^2 \tilde{N}^4 m_\nu^4 & \text{for } \tilde{N} \gg \frac{p}{m_\nu}. \end{cases} \quad (5.42)$$

Being proportional to the neutrino magnetic moment μ_ν the $SL\nu$ rate and power should be in general very small. However, several specific features of this phenomena might provide a believe in possible phenomenological consequences interesting for astrophysics. First of all, as it follows from Eqs. (5.37) and (5.38) the spatial distribution of the radiation is collimated along the neutrino propagation.

Secondly, for a wide range of matter densities the $SL\nu$ rate and power are increasing with the neutrino momentum. One may expect (Lobanov and Studenikin, 2003; Studenikin, 2004, 2007, 2006b; Lobanov and Studenikin, 2004; Grigoriev *et al.*, 2005a; Studenikin and Ternov, 2005; Grigorev *et al.*, 2005; Grigoriev *et al.*, 2005b, 2006b; Studenikin, 2007; Lobanov, 2005b,a; Studenikin, 2006a, 2008; Grigoriev *et al.*, 2009; Kuznetsov and Mikheev, 2006a; Grigoriev *et al.*, 2008, 2012b) that this radiation can be produced by high-energy neutrinos propagating in different astrophysical and cosmological environments.

For the average emitted photon energy

$$\langle \omega \rangle = I/\Gamma, \quad (5.43)$$

we obtain

$$\langle \omega \rangle \simeq \begin{cases} 2\tilde{N} \frac{p^2}{m_\nu} & \text{for } \tilde{N} \ll \frac{m_\nu}{p}, \\ \frac{1}{3} p & \text{for } \frac{m_\nu}{p} \ll \tilde{N} \ll \frac{p}{m_\nu}, \\ \tilde{N} m_\nu & \text{for } \tilde{N} \gg \frac{p}{m_\nu}. \end{cases} \quad (5.44)$$

Therefore, in the most interesting for possible astrophysical and cosmology applications case of ultra-high energy neutrinos, the average energy of the $SL\nu$ photons is one third of the neutrino momentum, so that in principle the $SL\nu$ spectrum spans up to the range peculiar of gamma-rays.

It should be emphasized that the $SL\nu$ mechanism of radiation (i.e. the transition between neutrino states with equal masses) can only become possible because of an external environment (plasma) influence on neutrino states. A possible impact of the background plasma on the $SL\nu$ radiation through the plasma influence on propagation of $SL\nu$ photons has been first considered in Studenikin and Ternov (2005); Grigorev *et al.* (2005); Grigoriev *et al.* (2005b, 2006b). The plasma effects for the $SL\nu$ were further studied in Kuznetsov and Mikheev (2006b, 2007) where the role of the $SL\nu$ plasmon mass was discussed. In the case of ultra-high energy neutrino (i.e., in the only case when the time scale of the process can be much less than the age of the Universe) the $SL\nu$ rate of Kuznetsov and Mikheev (2006b, 2007) exactly reproduces the result obtained in Studenikin and Ternov (2005); Grigorev *et al.* (2005); Grigoriev *et al.* (2005b, 2006b). For a more detailed discussion on the historical aspects of this issue see

Studenikin (2008); Grigoriev *et al.* (2006a); Kuznetsov and Mikheev (2006a); Grigoriev *et al.* (2008, 2009). The most complete and consistent study of the $SL\nu$ accounting for the plasma effects can be found in Grigoriev *et al.* (2012b).

Note that the $SL\nu$ process with transitions between neutrinos with different masses is considered in Grigoriev *et al.* (2012a). The $SL\nu$ mechanism in case of a neutrino moving in matter with simultaneous account for possible effects of Lorentz invariance violation is discussed in Kruglov (2014).

VI. INTERACTIONS WITH ELECTROMAGNETIC FIELDS

If neutrinos have non-trivial electromagnetic properties, they can interact with classical electromagnetic fields. Significant effects can occur, in particular, in neutrino astrophysics, since neutrinos can propagate over very long distances in astrophysical environments with magnetic fields. In this case even a very weak interaction can have large cumulative effects.

In Subsection VI.A we derive the effective potential of a neutrino propagating in a classical electromagnetic field. This potential can generate spin and spin flavor transitions, which are discussed in Subsection VI.B. We also review the limits on the effective neutrino magnetic moment obtained from analyses of solar neutrino data. In Subsection VI.C we discuss the modifications of neutrino magnetic moments in very strong magnetic fields. In Subsection VI.D we review the effects of a strong magnetic field on neutron decay. In Subsection VI.E we review neutrino-antineutrino pair production in a magnetic field and in Subsection VI.F we discuss neutrino-antineutrino pair production due to vacuum instability in a very strong magnetic field. In Subsection VI.G we review the energy quantization of neutrinos propagating in rotating media.

A. Effective potential

The coherent interactions of neutrinos with classical electromagnetic fields generate potentials which are similar to the matter potentials in Eq. (2.51) and must be taken into account in the study of flavor and spin evolution with an equation analogous to the MSW equation (2.54). This evolution in a magnetic field is discussed in detail in Section VI.B. Here we discuss the derivation of the effective neutrino potential in a classical electromagnetic field, which corresponds to the amplitude of coherent forward elastic scattering:

$$V_{h_i \rightarrow h_f} = \lim_{q \rightarrow 0} \frac{\langle \nu(p_f, h_f) | \int d^3x \mathcal{H}_{\text{em}}^{(\nu)}(x) | \nu(p_i, h_i) \rangle}{\langle \nu(p, h) | \nu(p, h) \rangle}, \quad (6.1)$$

where $q = p_i - p_f$ as above and the denominator enforces the correct normalization ($p = p_i = p_f$ in the limit $q \rightarrow 0$ and h is arbitrary). The interaction Hamiltonian $\mathcal{H}_{\text{em}}^{(\nu)}(x)$ is that in Eq. (3.2). Here we consider for simplicity only one neutrino (the generalization to more than one neutrino, with the possibility of coherent transitions between different massive neutrinos generated by transition form factors, is discussed later), allowing for possible helicity transitions ($h_f \neq h_i$), which are important in magnetic fields (see Section VI.B). Note that the hermiticity of $\mathcal{H}_{\text{em}}^{(\nu)}(x)$ implies that

$$V_{h_f \rightarrow h_i} = V_{h_i \rightarrow h_f}^*. \quad (6.2)$$

From the normalization of states in Eq. (A56) and Eqs. (3.2)–(3.8), we obtain

$$V_{h_i \rightarrow h_f} = \frac{1}{2E_\nu VT} \lim_{q \rightarrow 0} \overline{u^{(h_f)}(p_f)} \Lambda_\mu(q) u^{(h_i)}(p_i) \tilde{A}^\mu(q), \quad (6.3)$$

where T is the normalization time, $E_\nu = E_i = E_f$ in the limit $q \rightarrow 0$, and

$$\tilde{A}^\mu(q) = \int d^4x e^{-iq \cdot x} A^\mu(x) \quad (6.4)$$

is the Fourier transform of $A^\mu(x)$. Integrating by parts and neglecting an irrelevant surface term (which vanishes for well-behaved physical fields which vanish at infinity), we have

$$q_\alpha \tilde{A}^\mu(q) = -i \int d^4x e^{-iq \cdot x} \partial_\alpha A^\mu(x). \quad (6.5)$$

Using the expression (3.18) for $\Lambda_\mu(q)$, and the Gordon identity (A58) for the γ^μ term, we obtain, in the limit $q \rightarrow 0$,

$$\begin{aligned} V_{h_i \rightarrow h_f} = & \frac{1}{VT} \int d^4x \left[\mathfrak{q} \frac{p_\mu}{E_\nu} A^\mu(x) \delta_{h_f h_i} \right. \\ & + \frac{1}{4E_\nu} \overline{u^{(h_f)}(p)} \sigma_{\mu\nu} F^{\mu\nu}(x) \left(\frac{\mathfrak{q}}{2m} + \mathbb{I} + i\epsilon\gamma_5 \right) u^{(h_i)}(p) \\ & \left. - \frac{\mathfrak{a}}{2E_\nu} \overline{u^{(h_f)}(p)} j^\mu(x) \gamma_\mu \gamma_5 u^{(h_i)}(p) \right], \end{aligned} \quad (6.6)$$

where $p^\mu = p_i^\mu = p_f^\mu$. The electromagnetic tensor $F^{\mu\nu}(x)$ defined in Eq. (A70) contains the physical electric field $\vec{E}(x)$ and magnetic field $\vec{B}(x)$ (see Eq. (A72)).

Now, we take into account that propagating neutrinos are described by wave packets whose size is limited (see Giunti and Kim (2007)). Considering fields which are approximately constant over the extension of the neutrino wave packet, we can extract them from the integral in Eq. (6.6). Then, the integral simplifies with VT , leading to

$$\begin{aligned} V_{h_i \rightarrow h_f} = & \mathfrak{q} \frac{p_\mu}{E_\nu} A^\mu \delta_{h_f h_i} \\ & + \frac{1}{4E_\nu} \overline{u^{(h_f)}(p)} \sigma_{\mu\nu} F^{\mu\nu} \left(\frac{\mathfrak{q}}{2m} + \mathbb{I} + i\epsilon\gamma_5 \right) u^{(h_i)}(p) \\ & - \frac{\mathfrak{a}}{2E_\nu} \overline{u^{(h_f)}(p)} j^\mu \gamma_\mu \gamma_5 u^{(h_i)}(p). \end{aligned} \quad (6.7)$$

From Eq. (6.7) one can see that $V_{h_i \rightarrow h_f}$ depends on the four neutrino electromagnetic form factors at $q^2 = 0$, but the anapole moment contributes only in very special environments in which the medium is charged. Since we will discuss this special case in Section VII.C devoted to the anapole moment, in the following part of this Section we do not consider the anapole moment, assuming $j^\mu(x) = 0$.

Let us consider the first term in Eq. (6.7). In an electrostatic field $A^\mu = (A^0, 0, 0, 0)$, we have $V_{h_i \rightarrow h_f}^{(1)} = qA^0 \delta_{h_f h_i}$. This is the expected result, taking into account that A^0 is the electric potential. Of course this term can contribute to the neutrino potential only if neutrinos are millicharged particles (see Section VII.A).

Let us now consider the more interesting contribution of the second term in Eq. (6.7), which depends on the dipole magnetic and electric moments. Note that the charge generates a magnetic moment

$$\mu_q = \frac{q}{2m} = g \mu_{cl}^{(1/2)}, \quad \text{with } g = 2, \quad (6.8)$$

where $\mu_{cl}^{(s)} = qs/2m$ is the classical magnetic moment of a spin- s particle (see Jackson (1999)). This is the same magnetic moment obtained from the Dirac equation of a charged particle, with the well-known gyromagnetic ratio $g = 2$. For a normally-charged particle the additional contribution μ in Eq. (6.7) to the magnetic moment would be called “anomalous magnetic moment”, which is generated by an internal structure in the case of nucleons or by quantum loop corrections in the case of leptons (measured in the famous $g-2$ experiments). Since neutrinos are at most millicharged particles, the μ in Eq. (6.7) is traditionally called “magnetic moment”, and the possible contribution of μ_q is neglected. Moreover, μ_q does not contribute to helicity transitions, because it generates a spin precession which has the same frequency as the precession of the angular momentum generated by q (see Sakurai (1967)).

In the following, we study the effects of μ and ϵ assuming $q = 0$. We also wish to establish the connection of the neutrino potential with the classical potential for a non-relativistic particle (see Jackson (1999)),

$$V_{cl} = -\vec{\mu} \cdot \vec{B} - \vec{\epsilon} \cdot \vec{E}, \quad (6.9)$$

and the torque

$$\vec{T}_{cl} = \vec{\mu} \times \vec{B} + \vec{\epsilon} \times \vec{E}, \quad (6.10)$$

which generates the precession of the spin \vec{S} through $d\vec{S}/dt = \vec{T}_{cl}$.

Let us first consider the helicity-conserving potential $V_{h \rightarrow h}$. Using the method described in Appendix E, we obtain

$$V_{h \rightarrow h} = -\frac{m}{E_\nu} \left(\vec{\mu} \cdot \vec{B} + \vec{\epsilon} \cdot \vec{E} \right), \quad (6.11)$$

with

$$\vec{\mu} = h \frac{\vec{p}}{|\vec{p}|} \mu, \quad \vec{\epsilon} = h \frac{\vec{p}}{|\vec{p}|} \epsilon. \quad (6.12)$$

Hence, the helicity-conserving potential is proportional to the longitudinal components of the magnetic and electric fields. In the non-relativistic limit ($E_\nu \simeq m$) we obtain a potential which correspond to the classical one in Eq. (6.9). Note, however, that this potential is strongly suppressed by the small fraction m/E_ν for ultrarelativistic neutrinos in realistic experiments.

Considering now the helicity-flipping potential $V_{-h \rightarrow h}$, using the method described in Appendix E, if there is only an electric field \vec{E} , we obtain

$$V_{-h \rightarrow h}(\vec{E}) = \left(\epsilon + ih \frac{|\vec{p}|}{E_\nu} \mu \right) E_\perp, \quad (6.13)$$

where E_\perp is the transverse component of the electric field, i.e. that orthogonal to \vec{p} . In the case of a pure magnetic field \vec{B} , we have, with a similar notation,

$$V_{-h \rightarrow h}(\vec{B}) = \left(\mu - ih \frac{|\vec{p}|}{E_\nu} \epsilon \right) B_\perp. \quad (6.14)$$

where B_\perp is the transverse component of the magnetic field. The expression of $V_{-h \rightarrow h}$ in the general case of an electromagnetic field is given in Eq. (E11), from which one can see that in any case the helicity-flipping potential depends only on the transverse components of the electric and magnetic fields.

Notice that for non-relativistic neutrinos ($|\vec{p}| \ll E_\nu$) in practice $V_{-h \rightarrow h}(\vec{E})$ depends only on ϵ and $V_{-h \rightarrow h}(\vec{B})$ depends only on μ , as one may have expected:

$$V_{-h \rightarrow h}^{\text{nr}}(\vec{E}) \simeq \epsilon E_\perp = |\vec{\epsilon} \times \vec{E}|, \quad (6.15)$$

$$V_{-h \rightarrow h}^{\text{nr}}(\vec{B}) \simeq \mu B_\perp = |\vec{\mu} \times \vec{B}|. \quad (6.16)$$

Hence, in the non-relativistic limit the helicity-flipping potential corresponds to the classical torque in Eq. (6.10), which rotates the spin of the particle, causing periodic changes of the helicity.

The additional dependence of $V_{-h \rightarrow h}(\vec{E})$ on μ and that of $V_{-h \rightarrow h}(\vec{B})$ on ϵ for relativistic neutrinos are explained in Appendix E as a consequence of the relativistic transformations of the electric and magnetic fields and the correspondence of the electric and magnetic dipole moments with their classical counterparts only in the non-relativistic limit.

Let us finally consider the potential between different massive neutrinos, which is generated by transition electric and magnetic dipole moments,

$$V_{\nu_i^{(h_i)} \rightarrow \nu_f^{(h_f)}} = \lim_{q \rightarrow 0} \frac{\langle \nu_f(p_f, h_f) | \int d^3x \mathcal{H}_{\text{em}}^{(\nu)}(x) | \nu_i(p_i, h_i) \rangle}{\langle \nu(p, h) | \nu(p, h) \rangle}, \quad (6.17)$$

which is especially interesting for Majorana neutrinos which do not have diagonal electric and magnetic dipole moments. Here one can notice that it is impossible to have $p_i = p_f$ if $m_i \neq m_f$. However, we must remember that observable neutrinos are ultrarelativistic and their energy-momentum uncertainty is much larger than their mass differences (see Giunti and Kim (2007)). In this case, $\nu_i \rightarrow \nu_f$ transitions are possible in an electromagnetic field, as well as the coherent production of different massive neutrinos which is necessary for the oscillations discussed in Section II.D. In practice this means that in the calculation of V_{fi} we can approximate the neutrinos as massless. Under this approximation, the helicity-flipping potential in a transverse magnetic field in Eq. (6.14) can be generalized to

$$V_{\nu_i^{(-h)} \rightarrow \nu_f^{(h)}}(\vec{B}) = \left(\mu_{fi} - ih \frac{|\vec{p}|}{E_\nu} \epsilon_{fi} \right) B_\perp. \quad (6.18)$$

This potential is interesting because it determines the neutrino spin-flavor precession in a transverse magnetic field discussed in Section VI.B.

B. Spin-flavor precession

If neutrinos have magnetic moments, the spin can precess in a transverse magnetic field (Cisneros, 1971; Voloshin and Vysotsky, 1986; Okun *et al.*, 1986).

Let us first derive the spin precession of an ultrarelativistic Dirac neutrino generated by its diagonal magnetic moment μ . We consider a neutrino with four-momentum p which at the initial time $t = 0$ has a definite helicity h_i and is described by the state $|\nu(p, h_i)\rangle$. After propagation in a magnetic field \vec{B} , at the time t the neutrino is described by a superposition of both helicities:

$$|\nu(t)\rangle = \sum_{h=\pm 1} \psi_h(t) |\nu(p, h)\rangle. \quad (6.19)$$

The temporal evolution of $|\nu(t)\rangle$ is given by the Schrödinger equation

$$i \frac{d}{dt} |\nu(t)\rangle = \mathbb{H}_{\text{em}}(t) |\nu(t)\rangle, \quad (6.20)$$

where $|\nu(0)\rangle = |\nu(p, h_i)\rangle$ and $\mathbb{H}_{\text{em}}(t) = \int d^3x \mathcal{H}_{\text{em}}^{(\nu)}(x)$ is the effective interaction Hamiltonian, which can depend on time if the magnetic field is not constant. Here we neglect the irrelevant contribution of the vacuum Hamiltonian, which does not cause any change in helicity because the two helicity states have the same mass.

Multiplying Eq. (6.20) on the left by $\langle \nu(p, h) |$, we obtain the evolution equation for the helicity amplitudes

$$i \frac{d\psi_h(t)}{dt} = \sum_{h'=\pm 1} \psi_{h'}(t) V_{h' \rightarrow h}(t), \quad (6.21)$$

with the potential $V_{h' \rightarrow h}(t)$ given in Eq. (6.1) and $\psi_h(0) = \delta_{hh_i}$.

In Eq. (6.11) we have seen that the helicity-conserving potential, which depends on the longitudinal component of the magnetic field, is strongly suppressed for ultrarelativistic neutrinos. Hence, in practice only the transverse component of the magnetic field contributes through the helicity-flipping potential in Eq. (6.14). Considering for simplicity only the contribution of the magnetic moment μ , we have

$$V_{h' \rightarrow h}(t) = \mu B_\perp(t) \delta_{-hh'}. \quad (6.22)$$

Then, the evolution equation (6.21) can be written in the standard matrix form

$$i \frac{d}{dx} \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} = \begin{pmatrix} 0 & \mu B_\perp(x) \\ \mu B_\perp(x) & 0 \end{pmatrix} \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix}, \quad (6.23)$$

where we approximated the distance x along the neutrino trajectory with the time t for ultrarelativistic neutrinos and we adopted the standard notation $\psi_L \equiv \psi_{-1}$ and $\psi_R \equiv \psi_{+1}$ for the negative and positive helicity amplitudes of the left-handed and right-handed neutrinos, which are described, respectively, by the states $|\nu_L\rangle = |\nu(p, -1)\rangle$ and $|\nu_R\rangle = |\nu(p, +1)\rangle$. The differential equation (6.23) can be solved through the transformation

$$\begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \varphi_-(x) \\ \varphi_+(x) \end{pmatrix}. \quad (6.24)$$

The amplitudes $\varphi_-(x)$ and $\varphi_+(x)$ satisfy decoupled differential equations, whose solutions are

$$\varphi_\mp(x) = \exp \left[\pm i \int_0^x dx' \mu B_\perp(x') \right] \varphi_\mp(0). \quad (6.25)$$

If we consider an initial left-handed neutrino, we have

$$\begin{pmatrix} \psi_L(0) \\ \psi_R(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \varphi_-(0) \\ \varphi_+(0) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (6.26)$$

Then, the probability of $\nu_L \rightarrow \nu_R$ transitions is given by

$$P_{\nu_L \rightarrow \nu_R}(x) = |\psi_R(x)|^2 = \sin^2 \left(\int_0^x dx' \mu B_\perp(x') \right). \quad (6.27)$$

Note that the transition probability is independent from the neutrino energy (contrary to the case of flavor oscillations) and the amplitude of the oscillating probability is unity. Hence, when the argument of the sine is equal to $\pi/2$ there is complete $\nu_L \rightarrow \nu_R$ conversion.

The precession $\nu_{eL} \rightarrow \nu_{eR}$ in the magnetic field of the Sun was considered in 1971 (Cisneros, 1971) as a possible solution of the solar neutrino problem. If neutrinos are Dirac particles, right-handed neutrinos are sterile and a $\nu_{eL} \rightarrow \nu_{eR}$ conversion could explain the disappearance of active solar ν_{eL} 's.

In 1986 it was realized (Voloshin and Vysotsky, 1986; Okun *et al.*, 1986) that the matter effect during neutrino propagation inside of the Sun suppresses $\nu_{eL} \rightarrow \nu_{eR}$ transition by lifting the degeneracy of ν_{eL} and ν_{eR} (see also Barbieri and Fiorentini (1988)). Indeed, taking into account matter effects, the evolution equation (6.23) becomes

$$i \frac{d}{dx} \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} = \begin{pmatrix} V(x) & 2\mu B_\perp(x) \\ \mu B_\perp(x) & 0 \end{pmatrix} \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix}, \quad (6.28)$$

with the appropriate potential V which depends on the neutrino flavor, according to Eq. (2.50). In the case of a constant matter density, this differential equation can be solved analytically with the orthogonal transformation

$$\begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} = \begin{pmatrix} \cos \xi & \sin \xi \\ -\sin \xi & \cos \xi \end{pmatrix} \begin{pmatrix} \varphi_-(x) \\ \varphi_+(x) \end{pmatrix}. \quad (6.29)$$

The angle ξ is chosen in order to diagonalize the matrix operator in Eq. (6.28):

$$\sin 2\xi = \frac{2\mu B_\perp}{\Delta E_M}, \quad (6.30)$$

with the effective energy splitting in matter

$$\Delta E_M = \sqrt{V^2 + (2\mu B_\perp)^2}. \quad (6.31)$$

The decoupled evolution of $\varphi_\mp(x)$ is given by

$$\varphi_\mp(x) = \exp \left[-\frac{i}{2} (V \mp \Delta E_M) x \right] \varphi_\mp(0). \quad (6.32)$$

Considering an initial left-handed neutrino,

$$\begin{pmatrix} \varphi_-(0) \\ \varphi_+(0) \end{pmatrix} = \begin{pmatrix} \cos \xi \\ \sin \xi \end{pmatrix}, \quad (6.33)$$

we obtain the oscillatory transition probability

$$P_{\nu_L \rightarrow \nu_R}(x) = |\psi_R(x)|^2 = \sin^2 2\xi \sin^2 \left(\frac{1}{2} \Delta E_M x \right). \quad (6.34)$$

Since in matter $\Delta E_M > 2\mu B_\perp$, the matter effect suppresses the amplitude of $\nu_L \rightarrow \nu_R$ transitions. However, these transitions are still independent from the neutrino energy, which does not enter in the evolution equation (6.28).

When it was known, in 1986 (Voloshin and Vysotsky, 1986; Okun *et al.*, 1986), that the matter potential has the effect of suppressing $\nu_L \rightarrow \nu_R$ transitions because it breaks the degeneracy of left-handed and right-handed states, it did not take long to realize, in 1988 (Akhmedov, 1988; Lim and Marciano, 1988), that the matter potentials can cause resonant spin-flavor precession if different flavor neutrinos have transition magnetic moments (spin-flavor precession in vacuum was previously

discussed by Schechter and Valle (1981)). The application of this mechanism to solar neutrinos has been discussed in the following years by many authors (Minakata and Nunokawa, 1989; Akhmedov and Bychuk, 1989; Balantekin *et al.*, 1990; Raghavan *et al.*, 1991; Akhmedov, 1991b,a; Pulido, 1992b; Shi *et al.*, 1993; Balantekin and Loreti, 1992; Akhmedov *et al.*, 1993a,c, 1995; Pulido and Akhmedov, 2000; Akhmedov and Pulido, 2000; Dev and Sharma, 2000; Chauhan and Pulido, 2002; Barranco *et al.*, 2002; Akhmedov and Pulido, 2003; Chauhan *et al.*, 2003; Miranda *et al.*, 2004a,b; Balantekin and Volpe, 2005; Chauhan and Pulido, 2004; Chauhan *et al.*, 2005; Pulido *et al.*, 2005; Guzzo *et al.*, 2005; Friedland, 2005; Chauhan *et al.*, 2007; Picariello *et al.*, 2007; Yilmaz, 2008; Raffelt and Rashba, 2010; Das *et al.*, 2009; Guzzo *et al.*, 2012).

Let us consider a neutrino state which is a superposition of different massive neutrinos with both helicities:

$$|\nu(t)\rangle = \sum_k \sum_{h=\pm 1} \psi_{kh}(t) |\nu_k(p, h)\rangle, \quad (6.35)$$

where $\psi_{kh}(t)$ is the amplitude of the neutrino with mass m_k and helicity h . The temporal evolution of $|\nu(t)\rangle$ is given by the Schrödinger equation

$$i \frac{d}{dt} |\nu(t)\rangle = \mathbb{H}(t) |\nu(t)\rangle, \quad (6.36)$$

with the initial condition $|\nu(0)\rangle = |\nu_i(p, h_i)\rangle$. Multiplying the evolution equation on the left by $\langle \nu_k(p, h) |$, we obtain the evolution equation for the helicity amplitudes of the different massive neutrinos

$$i \frac{d\psi_{kh}(t)}{dt} = \sum_j \sum_{h'=\pm 1} \frac{\langle \nu_k(p, h) | \mathbb{H}(t) | \nu_j(p, h') \rangle}{\langle \nu(p, h) | \nu(p, h) \rangle} \psi_{jh'}(t), \quad (6.37)$$

with $\psi_{kh}(0) = \delta_{hh_i} \delta_{ki}$. The effective Hamiltonian $\mathbb{H}(t)$ is the sum of a vacuum Hamiltonian \mathbb{H}_0 , a weak interaction Hamiltonian $\mathbb{H}_w(t)$ which generates the effective potentials (2.50) of flavor neutrinos in matter, and the electromagnetic Hamiltonian $\mathbb{H}_{em}(t)$ already considered in Eq. (6.20). For ultrarelativistic neutrinos,

$$\frac{\langle \nu_k(p, h) | \mathbb{H}_0 | \nu_j(p, h') \rangle}{\langle \nu(p, h) | \nu(p, h) \rangle} = \left(E_\nu + \frac{m_k^2}{2E_\nu} \right) \delta_{kj} \delta_{hh'}, \quad (6.38)$$

where E_ν is the neutrino energy neglecting mass contributions.

In order to calculate the matrix element of $\mathbb{H}_w(t)$, we must take into account the mixing of neutrino states in Eq. (2.35), which applies to left-handed neutrinos:

$$|\nu_k(p, -)\rangle = \sum_\alpha U_{\alpha k} |\nu_\alpha(p, -)\rangle. \quad (6.39)$$

For right-handed Dirac neutrinos the mixing is arbitrary, because right-handed Dirac neutrinos are sterile to weak

interactions. On the other hand, since right-handed Majorana neutrinos interact as right-handed Dirac antineutrinos, their mixing is given by

$$|\nu_k^M(p, +)\rangle = \sum_{\alpha} U_{\alpha k}^* |\nu_{\alpha}(p, +)\rangle. \quad (6.40)$$

Therefore, we define the generalized mixing relation

$$|\nu_k(p, h)\rangle = \sum_{\alpha} U_{\alpha k}^{(h)} |\nu_{\alpha}(p, h)\rangle, \quad (6.41)$$

with $U^{(-)} = U$ and

$$\text{Dirac : } U^{(+)} = U; \quad (6.42)$$

$$\text{Majorana : } U^{(+)} = U^*. \quad (6.43)$$

The arbitrary choice for Dirac neutrinos has been made for simple convenience. Then, for the matrix element of $\mathbb{H}_w(t)$ we obtain

$$\frac{\langle \nu_k(p, h) | \mathbb{H}_w(t) | \nu_j(p, h') \rangle}{\langle \nu(p, h) | \nu(p, h) \rangle} = \sum_{\alpha} U_{\alpha k}^{(h)*} U_{\alpha j}^{(h)} V_{\alpha}^{(h)}(t) \delta_{hh'}, \quad (6.44)$$

where $V_{\alpha}^{(-)} = V_{\alpha}$, with the potential V_{α} in Eq. (2.50), and

$$\text{Dirac : } V_{\alpha}^{(+)} = 0; \quad (6.45)$$

$$\text{Majorana : } V_{\alpha}^{(+)} = -V_{\alpha}. \quad (6.46)$$

As remarked before Eq. (6.22), the helicity-conserving potential generated by $\mathbb{H}_{\text{em}}^{(\nu)}(t)$, which depends on the longitudinal component of the magnetic field, is strongly suppressed for ultrarelativistic neutrinos. Then, from Eq. (6.18), considering for simplicity only the contribution of the magnetic moments, we have

$$\frac{\langle \nu_k(p, h) | \mathbb{H}_{\text{em}}^{(\nu)}(t) | \nu_j(p, h') \rangle}{\langle \nu(p, h) | \nu(p, h) \rangle} = \mathbb{J}_{kj} B_{\perp}(t) \delta_{-hh'}. \quad (6.47)$$

Plugging Eqs. (6.38), (6.44) and (6.47) in Eq. (6.37), neglecting the irrelevant common energy contribution in Eq. (6.38) and approximating the distance x along the neutrino trajectory with the time t for ultrarelativistic neutrinos, one obtains the evolution equations of the helicity amplitudes of the different massive neutrinos:

$$\begin{aligned} i \frac{d\psi_{k,h}(x)}{dx} = & \sum_j \sum_{h'=\pm 1} \left[\left(\frac{m_k^2}{2E_{\nu}} \delta_{kj} \right. \right. \\ & + \sum_{\alpha} U_{\alpha k}^{(h)*} V_{\alpha}^{(h)}(x) U_{\alpha j}^{(h)} \Big) \delta_{hh'} \\ & \left. \left. + \mathbb{J}_{kj} B_{\perp}(x) \delta_{-hh'} \right] \psi_{j,h'}(x), \end{aligned} \quad (6.48)$$

In order to study flavor and helicity transitions, it is more convenient to work in the flavor basis. Using the mixing

of neutrino states in Eq. (2.34), the state (6.35) with $t = x$ can be written as

$$|\nu(x)\rangle = \sum_{\alpha} \sum_{h=\pm 1} \psi_{\alpha,h}(x) |\nu_{\alpha}(p, h)\rangle, \quad (6.49)$$

with the flavor and helicity amplitudes

$$\psi_{\alpha,h}(x) = \sum_k U_{\alpha k}^{(h)} \psi_{k,h}(x), \quad (6.50)$$

which obey the evolution equation

$$\begin{aligned} i \frac{d\psi_{\alpha,h}(x)}{dx} = & \sum_{\beta} \sum_{h'=\pm 1} \left[\left(\sum_k U_{\alpha k}^{(h)} \frac{m_k^2}{2E_{\nu}} U_{\beta k}^{(h)*} \right. \right. \\ & \left. \left. + V_{\alpha}^{(h)}(x) \delta_{\alpha\beta} \right) \delta_{hh'} \right. \\ & \left. + \mathbb{J}_{\alpha\beta}^{(h,h')} B_{\perp}(x) \delta_{-hh'} \right] \psi_{\beta,h'}(x), \end{aligned} \quad (6.51)$$

with the effective magnetic moments in the flavor basis

$$\mathbb{J}_{\alpha\beta}^{(h,h')} = \sum_{k,j} U_{\alpha k}^{(h)} \mathbb{J}_{kj} U_{\beta j}^{(h')*}. \quad (6.52)$$

For Dirac neutrinos, from Eq. (6.42) we have

$$\mathbb{J}_{\alpha\beta}^{(-,+)} = \mathbb{J}_{\alpha\beta}^{(+,-)} = \sum_{k,j} U_{\alpha k} \mathbb{J}_{kj} U_{\beta j}^* \equiv \mathbb{J}_{\alpha\beta}. \quad (6.53)$$

Then, from Eq. (3.36) we obtain

$$\mathbb{J}_{jk} = \mathbb{J}_{kj}^* \Rightarrow \mathbb{J}_{\beta\alpha} = \mathbb{J}_{\alpha\beta}^*. \quad (6.54)$$

For Majorana neutrinos, from Eq. (6.43) we have

$$\mathbb{J}_{\alpha\beta}^{(-,+)} = \sum_{k,j} U_{\alpha k} \mathbb{J}_{kj} U_{\beta j}, \quad (6.55)$$

$$\mathbb{J}_{\alpha\beta}^{(+,-)} = \sum_{k,j} U_{\alpha k}^* \mathbb{J}_{kj} U_{\beta j}^*. \quad (6.56)$$

From Eqs. (3.67) and (3.69) it follows that for Majorana neutrinos the matrix of magnetic moments is antisymmetric and the transition magnetic moments are imaginary:

$$\mathbb{J}_{jk} = -\mathbb{J}_{kj} = \mathbb{J}_{kj}^*. \quad (6.57)$$

The antisymmetric property is preserved in the flavor basis:

$$\mathbb{J}_{\alpha\beta}^{(-,+)} = -\mathbb{J}_{\beta\alpha}^{(-,+)}, \quad \mathbb{J}_{\alpha\beta}^{(+,-)} = -\mathbb{J}_{\beta\alpha}^{(+,-)}. \quad (6.58)$$

Hence, there are no diagonal magnetic moments in the flavor basis as in the mass basis. Moreover, we have

$$\mathbb{J}_{\alpha\beta}^{(+,-)} = -\mathbb{J}_{\alpha\beta}^{(-,+)*}. \quad (6.59)$$

In the following we discuss the spin-flavor evolution equation in the two-neutrino mixing approximation,

which is interesting for understanding the relevant features of neutrino spin-flavor precession. Having in mind the application to solar neutrinos, we consider the ν_e - ν_a mixing discussed in Subsection II.D, where ν_a is the linear combination of ν_μ and ν_τ in Eq. (2.68). Neglecting the small effects due to ϑ_{13} , we have

$$\begin{pmatrix} \psi_{e,h}(x) \\ \psi_{a,h}(x) \end{pmatrix} = R_{12} \begin{pmatrix} \psi_{1,h}(x) \\ \psi_{2,h}(x) \end{pmatrix}, \quad (6.60)$$

with

$$R_{12} = \begin{pmatrix} \cos \vartheta_{12} & \sin \vartheta_{12} \\ -\sin \vartheta_{12} & \cos \vartheta_{12} \end{pmatrix}. \quad (6.61)$$

Considering Dirac neutrinos, from Eq. (6.51) it follows that the generalization of Eq. (6.23) to two-neutrino ν_e - ν_a mixing is, using the analogous notation $\psi_{\alpha L} \equiv \psi_{\alpha,-1}$ and $\psi_{\alpha R} \equiv \psi_{\alpha,+1}$,

$$i \frac{d}{dx} \begin{pmatrix} \psi_{eL}(x) \\ \psi_{aL}(x) \\ \psi_{eR}(x) \\ \psi_{aR}(x) \end{pmatrix} = H \begin{pmatrix} \psi_{eL}(x) \\ \psi_{aL}(x) \\ \psi_{eR}(x) \\ \psi_{aR}(x) \end{pmatrix}, \quad (6.62)$$

with the effective Hamiltonian matrix

$$H = \begin{pmatrix} -\frac{\Delta m^2}{4E_\nu} \cos 2\vartheta_{12} + V_e & \frac{\Delta m^2}{4E_\nu} \sin 2\vartheta_{12} & \mathbb{P}_{ee} B_\perp(x) & \mathbb{P}_{ea} B_\perp(x) \\ \frac{\Delta m^2}{4E_\nu} \sin 2\vartheta_{12} & \frac{\Delta m^2}{4E_\nu} \cos 2\vartheta_{12} + V_a & \mathbb{P}_{ea}^* B_\perp(x) & \mathbb{P}_{aa} B_\perp(x) \\ \mathbb{P}_{ee} B_\perp(x) & \mathbb{P}_{ea} B_\perp(x) & -\frac{\Delta m^2}{4E_\nu} \cos 2\vartheta_{12} & \frac{\Delta m^2}{4E_\nu} \sin 2\vartheta_{12} \\ \mathbb{P}_{ea}^* B_\perp(x) & \mathbb{P}_{aa} B_\perp(x) & \frac{\Delta m^2}{4E_\nu} \sin 2\vartheta_{12} & \frac{\Delta m^2}{4E_\nu} \cos 2\vartheta_{12} \end{pmatrix}, \quad (6.63)$$

with the effective magnetic moments in the flavor basis given by

$$\begin{pmatrix} \mathbb{P}_{ee} & \mathbb{P}_{ea} \\ \mathbb{P}_{ea}^* & \mathbb{P}_{aa} \end{pmatrix} = R_{12} \begin{pmatrix} \mathbb{P}_{11} & \mathbb{P}_{12} \\ \mathbb{P}_{12}^* & \mathbb{P}_{22} \end{pmatrix} R_{12}^T. \quad (6.64)$$

The matter potential can generate resonances, which occur when two diagonal elements of H become equal. Besides the standard MSW resonance in the $\nu_{eL} \leftrightarrow \nu_{aL}$ channel discussed in Subsection II.D, there are two possibilities:

1. There is a resonance in the $\nu_{eL} \leftrightarrow \nu_{aR}$ channel for

$$V_e = \frac{\Delta m^2}{2E_\nu} \cos 2\vartheta_{12}. \quad (6.65)$$

The density at which this resonance occurs is not the same as that of the MSW resonance, given by Eq. (2.64), because of the neutral-current contribution to $V_e = V_{CC} + V_{NC}$. The location of this resonance depends on both N_e and N_n .

2. There is a resonance in the $\nu_{aL} \leftrightarrow \nu_{eR}$ channel for

$$V_a = -\frac{\Delta m^2}{2E_\nu} \cos 2\vartheta_{12}. \quad (6.66)$$

If $\cos 2\vartheta_{12} > 0$, this resonance is possible in normal matter, since the sign of $V_a = V_{NC}$ is negative, as one can see from Eq. (2.51).

In practice the effect of these resonances could be the disappearance of active ν_{eL} or ν_{aL} into sterile right-handed states.

Let us consider now the case of Majorana neutrinos. The evolution equation of the amplitudes is given by Eq. (6.62) with the effective Hamiltonian matrix

$$H = \begin{pmatrix} -\frac{\Delta m^2}{4E_\nu} \cos 2\vartheta_{12} + V_e & \frac{\Delta m^2}{4E_\nu} \sin 2\vartheta_{12} & 0 & \mathbb{P}_{ea} B_\perp(x) \\ \frac{\Delta m^2}{4E_\nu} \sin 2\vartheta_{12} & \frac{\Delta m^2}{4E_\nu} \cos 2\vartheta_{12} + V_a & -\mathbb{P}_{ea} B_\perp(x) & 0 \\ 0 & -\mathbb{P}_{ea}^* B_\perp(x) & -\frac{\Delta m^2}{4E_\nu} \cos 2\vartheta_{12} - V_e & \frac{\Delta m^2}{4E_\nu} \sin 2\vartheta_{12} \\ \mathbb{P}_{ea}^* B_\perp(x) & 0 & \frac{\Delta m^2}{4E_\nu} \sin 2\vartheta_{12} & \frac{\Delta m^2}{4E_\nu} \cos 2\vartheta_{12} - V_a \end{pmatrix}, \quad (6.67)$$

with

$$\mathbb{P}_{ea} \equiv \mathbb{P}_{ea}^{(-,+)} = \mathbb{P}_{12} e^{i\lambda_{12}}, \quad (6.68)$$

where λ_{12} is the Majorana phase in Eq. (2.29).

As in the Dirac case, there are two possible resonances besides the standard MSW resonance in the $\nu_{eL} \rightleftharpoons \nu_{aL}$ channel:

1. There is a resonance in the $\nu_{eL} \rightleftharpoons \nu_{aR}$ channel for

$$V_{CC} + 2V_{NC} = \frac{\Delta m^2}{2E_\nu} \cos 2\vartheta_{12}. \quad (6.69)$$

2. There is a resonance in the $\nu_{aL} \rightleftharpoons \nu_{eR}$ channel for

$$V_{CC} + 2V_{NC} = -\frac{\Delta m^2}{2E_\nu} \cos 2\vartheta_{12}. \quad (6.70)$$

The location of both resonances depend on both N_e and N_n . If $\cos 2\vartheta_{12} > 0$, only the first resonance can occur in normal matter, where $N_n \simeq N_e/6$. A realization of the second resonance requires a large neutron number density, as that in a neutron star.

The neutrino spin oscillations in a transverse magnetic field with a possible rotation of the field-strength vector in a plane orthogonal to the neutrino-propagation direction (such rotating fields may exist in the convective zone of the Sun) have been considered in Vidal and Wudka (1990); Smirnov (1991); Akhmedov *et al.* (1993b); Likhachev and Studenikin (1995). The effect of the magnetic-field rotation may substantially shift the resonance point of neutrino oscillations. Neutrino spin oscillations in electromagnetic fields with other different configurations, including a longitudinal magnetic field and the field of an electromagnetic wave, were examined in Akhmedov and Khlopov (1988b,a) and Egorov *et al.* (2000); Lobanov and Studenikin (2001); Dvornikov and Studenikin (2001, 2004c).

It is possible to formulate a criterion (Likhachev and Studenikin, 1995) for finding out if the neutrino spin and spin-flavor precession is significant for given neutrino and background medium properties. The probability of oscillatory transitions between two neutrino states $\nu_{\alpha L} \rightleftharpoons \nu_{\beta R}$ can be expressed in terms of the elements of the effective Hamiltonian matrices (6.63) and (6.67) as

$$P_{\nu_{\alpha L} \rightleftharpoons \nu_{\beta R}} = \sin^2 \vartheta_{\text{eff}} \sin^2 \frac{x\pi}{L_{\text{eff}}}, \quad (6.71)$$

where

$$\sin^2 \vartheta_{\text{eff}} = \frac{4H_{\alpha\beta}^2}{4H_{\alpha\beta}^2 + (H_{\beta\beta} - H_{\alpha\alpha})^2}, \quad (6.72)$$

$$L_{\text{eff}} = \frac{2\pi}{\sqrt{4H_{\alpha\beta}^2 + (H_{\beta\beta} - H_{\alpha\alpha})^2}}. \quad (6.73)$$

The transition probability can be of order unity if the following two conditions hold simultaneously: 1) the amplitude of the transition probability must be sizable (at least $\sin^2 \vartheta_{\text{eff}} \gtrsim 1/2$); 2) the neutrino path length in a medium with a magnetic field should be longer than half

the effective length of oscillations L_{eff} . In accordance with this criterion, it is possible to introduce the critical strength of a magnetic field B_{cr} which determines the region of field values $B_\perp > B_{\text{cr}}$ at which the probability amplitude is not small ($\sin^2 \vartheta_{\text{eff}} > 1/2$):

$$B_{\text{cr}} = \frac{1}{2\mu_{\beta\alpha}} \sqrt{(H_{\beta\beta} - H_{\alpha\alpha})^2}. \quad (6.74)$$

Consider, for instance, the case of $\nu_{eL} \rightleftharpoons \nu_{aR}$ transitions of Majorana neutrinos. From Eqs. (6.67) and (6.74), it follows (Likhachev and Studenikin, 1995) that

$$B_{\text{cr}} = \left| \frac{1}{2\mu_{ae}} \left(\frac{\Delta m^2 \cos 2\vartheta_{12}}{2E_\nu} - \sqrt{2} G_F N_{\text{eff}} \right) \right|, \quad (6.75)$$

where $N_{\text{eff}} = N_e - N_n$. For getting numerical estimates of B_{cr} it is convenient to re-write Eq. (6.75) in the following form:

$$B_{\text{cr}} \approx 43 \frac{\mu_B}{\mu_{ae}} \left| A \left(\frac{\Delta m^2}{\text{eV}^2} \right) \left(\frac{\text{MeV}}{E_\nu} \right) - 2.5 \times 10^{-31} \left(\frac{N_{\text{eff}}}{\text{cm}^{-3}} \right) \right| \text{G}. \quad (6.76)$$

An interesting feature of the evolution equation (6.62) in the case of Majorana neutrinos is that the interplay of spin precession and flavor oscillations can generate $\nu_{eL} \rightarrow \nu_{eR}$ transitions (Akhmedov, 1991a). Since ν_{eR} interacts as right-handed Dirac antineutrinos, it is often denoted by $\bar{\nu}_{eR}$, or only $\bar{\nu}_e$, and called “electron antineutrino”. This state can be detected through the inverse β -decay reaction

$$\bar{\nu}_e + p \rightarrow n + e^+, \quad (6.77)$$

having a threshold $E_{\text{th}} = 1.8 \text{ MeV}$.

The possibility of $\nu_{eL} \rightarrow \bar{\nu}_{eR}$ transitions generated by spin-flavor precession of Majorana neutrinos is particularly interesting for solar neutrinos, which experience matter effects in the interior of the Sun in the presence of the solar magnetic field (see Pulido (1992b); Shi *et al.* (1993)). Taking into account the dominant $\nu_e \rightarrow \nu_a$ transitions due to neutrino oscillations, with ν_a given by Eq. (2.68), the probability of solar $\nu_{eL} \rightarrow \bar{\nu}_{eR}$ transitions is given by Akhmedov and Pulido (2003)

$$P_{\nu_{eL} \rightarrow \bar{\nu}_{eR}} \simeq 1.8 \times 10^{-10} \sin^2 2\vartheta_{12} \times \left(\frac{\mu_{ea}}{10^{-12} \mu_B} \frac{B_\perp (0.05 R_\odot)}{10 \text{ kG}} \right)^2, \quad (6.78)$$

where μ_{ea} is the transition magnetic moment in Eq. (6.68), R_\odot is the radius of the Sun, and the values of ϑ_{12} and ϑ_{23} in Tab. II.

It is also possible that spin-flavor precession occurs in the convective zone of the Sun, where there can be random turbulent magnetic fields (Miranda *et al.*, 2004a,b;

Friedland, 2005). In this case (Raffelt and Rashba, 2010),

$$P_{\nu_{eL} \rightarrow \bar{\nu}_{eR}} \approx 10^{-7} S^2 \left(\frac{\mu_{ea}}{10^{-12} \mu_B} \right)^2 \left(\frac{B}{20 \text{ kG}} \right)^2 \times \left(\frac{3 \times 10^4 \text{ km}}{L_{\max}} \right)^{p-1} \left(\frac{8 \times 10^{-5} \text{ eV}^2}{\Delta m_S^2} \right)^p \times \left(\frac{E_\nu}{10 \text{ MeV}} \right)^p \left(\frac{\cos^2 \vartheta_{12}}{0.7} \right)^p, \quad (6.79)$$

where S is a factor of order unity depending on the spatial configuration of the magnetic field, B is the average strength of the magnetic field at the spatial scale L_{\max} , which is the largest scale of the turbulence, p is the power of the turbulence scaling, Δm_S^2 is the solar neutrino squared-mass difference in Tab. II, and E_ν is the neutrino energy. A possible value of p is $5/3$ (Miranda *et al.*, 2004a,b; Friedland, 2005), corresponding to Kolmogorov turbulence. Conservative values for the other parameters are $B = 20 \text{ kG}$ and $L_{\max} = 3 \times 10^4 \text{ km}$.

In 2002, the Super-Kamiokande Collaboration established for the flux of solar $\bar{\nu}_e$'s a 90% CL an upper limit of 0.8% of the Standard Solar Model (SSM) neutrino flux in the range of energy from 8 to 20 MeV (Gando *et al.*, 2003) by taking as a reference the BP00 SSM prediction $\phi_{\text{SB}}^{\text{BP00}} = 5.05 \times 10^6 \text{ cm}^{-2} \text{ s}^{-1}$ for the solar ^8B flux (Bahcall *et al.*, 2001) and assuming an undistorted ^8B spectrum for the $\bar{\nu}_e$'s. This limit was improved in 2003 by the KamLAND Collaboration (Eguchi *et al.*, 2004) to 2.8×10^{-4} of the BP00 SSM prediction at 90% CL by measuring $\phi_{\bar{\nu}_e} < 370 \text{ cm}^{-2} \text{ s}^{-1}$ (90% CL) in the energy range 8.3 – 14.8 MeV, which corresponds to $\phi_{\bar{\nu}_e} < 1250 \text{ cm}^{-2} \text{ s}^{-1}$ (90% CL) in the entire ^8B energy range assuming an undistorted spectrum.

Recently, the Borexino collaboration established the best limit on the probability of solar $\nu_{eL} \rightarrow \bar{\nu}_{eR}$ transitions (Bellini, 2011),

$$P_{\nu_{eL} \rightarrow \bar{\nu}_{eR}} < 1.3 \times 10^{-4} \quad (90\% \text{ CL}), \quad (6.80)$$

by taking as a reference $\phi_{\text{SB}}^{\text{SSM}} = 5.88 \times 10^6 \text{ cm}^{-2} \text{ s}^{-1}$ (Serenelli *et al.*, 2009) and assuming an undistorted ^8B spectrum for the $\bar{\nu}_e$'s. They measured $\phi_{\bar{\nu}_e} < 320 \text{ cm}^{-2} \text{ s}^{-1}$ (90% CL) for $E_{\bar{\nu}_e} > 7.3 \text{ MeV}$, which corresponds to $\phi_{\bar{\nu}_e} < 760 \text{ cm}^{-2} \text{ s}^{-1}$ (90% CL) in the entire ^8B energy range assuming an undistorted spectrum.

The implications of the limits on the flux of solar $\bar{\nu}_e$'s on Earth for the spin-flavor precession of solar neutrinos have been studied in several papers (Akhmedov and Pulido, 2003; Chauhan *et al.*, 2003; Miranda *et al.*, 2004a,b; Balantekin and Volpe, 2005; Guzzo *et al.*, 2005; Friedland, 2005; Yilmaz, 2008), taking into account the dominant $\nu_e \rightarrow \nu_\mu, \nu_\tau$ transitions due to neutrino oscillations (see Giunti and Kim (2007); Gonzalez-Garcia and Maltoni (2008); Bilenky (2010); Xing and Zhou (2011)). Using Eqs. (6.78) and (6.80), we obtain

$$\mu_{ea} \lesssim 1.3 \times 10^{-12} \frac{7 \text{ MG}}{B_\perp (0.05 R_\odot)} \mu_B, \quad (6.81)$$

with $600 \text{ G} \lesssim B_\perp (0.05 R_\odot) \lesssim 7 \text{ MG}$ (Bellini, 2011). In the case of spin-flavor precession in the convective zone of the Sun with random turbulent magnetic fields, Eqs. (6.79) and (6.80) give, assuming $p = 5/3$,

$$\mu_{ea} \lesssim 4 \times 10^{-11} S^{-1} \frac{20 \text{ kG}}{B} \left(\frac{L_{\max}}{3 \times 10^4 \text{ km}} \right)^{1/3} \mu_B. \quad (6.82)$$

The spin-flavor mechanism was also considered (Pulido *et al.*, 2005) in order to describe time variations of solar-neutrino fluxes in Gallium experiments. The effect of a nonzero neutrino magnetic moment is also of interest in connection with the analysis of helioseismological observations (Couvidat *et al.*, 2003).

The idea that the neutrino magnetic moment may solve the supernova problem, i.e. that the neutrino spin-flip transitions in a magnetic field provide an efficient mechanism of energy transfer from a protoneutron star, was first discussed in Dar (1987) and then investigated in some detail in Nussinov and Rephaeli (1987); Goldman *et al.* (1988); Lattimer and Cooperstein (1988); Barbieri and Mohapatra (1988). The possibility of a loss of up to half of the active left-handed neutrinos because of their transition to sterile right-handed neutrinos in strong magnetic fields at the boundary of the neutron star (the so-called boundary effect) was considered in Likhachev and Studenikin (1995).

C. Mass operator in a magnetic field

The external electromagnetic fields influence on neutrinos, in addition to their *direct influence* on neutrinos through non-trivial neutrino electromagnetic properties, can show up due to fields effects on the real and virtual electrically charged particles (and also particles with other non-vanishing electromagnetic properties) interacting with neutrinos. These types of effects correspond to the *indirect influence* of the external electromagnetic fields on neutrinos and can be also considered as manifestation of neutrinos electromagnetic properties.

In this and next two sections we consider three examples of the mentioned above effects of the external electromagnetic fields *indirect influence* on neutrinos that are believed to have important consequences for neutrinos propagating in various astrophysical and cosmology settings: 1) the modification of the neutrino mass operator in the external electromagnetic field and shift of the neutrino electromagnetic properties, in particular, neutrino magnetic moment in external magnetic field, 2) the beta decay (anti)neutrinos spacial distribution asymmetry due to the magnetic field influence on the electron and proton, and 3) processes of neutrinos interaction with real particles, such as the neutrino pair synchrotron emission by an electron, that could only become possible under the influence of external electromagnetic fields.

The proceeding discussion on the neutrino electromagnetic properties is based on the one-photon approach to description of a neutrino response to an electromagnetic field presence. As a matter of fact, this approach is appropriate to the case when the strength of an external field is not too high. However, in case of very strong fields the one-photon approach could give not complete vision on how a neutrino react on the electromagnetic field presence. One would reasonably expect to come across this situation, for instance, in the vicinity of a neutron star where very strong magnetic fields are believed to exist. In the latter case one has to consider the neutrino electromagnetic properties within the multi-photon approach. If one still would use the introduces above neutrino form factors and the corresponding multipole moments then possibility of their dependence on the external field should be admitted.

In this Section we discuss, as one representative example, the dependence of a neutrino magnetic moment on the external magnetic field strength. This effect can be considered as a result of indirect influence of a magnetic field on the neutrino due to the field action on virtual charged particles that run in loop diagrams contributing to the neutrino magnetic moment.

The neutrino magnetic moment dependence on the magnetic field strength was investigated for the first time in a series of papers by Borisov *et al.* (1985, 1987, 1988, 1989a) where the self-energy of a neutrino (without account for neutrino mixing) in presence of an arbitrary electromagnetic field was considered¹⁰. We use the obtained results of Borisov *et al.* (1985) for evaluation of the massive neutrino magnetic moment dependence on external magnetic field with an appropriate generalization aimed to account for the neutrino mixing.

Evaluation of the neutrino magnetic moment dependence on the magnetic field is based on the most general approach to the problem of a lepton motion in an external electromagnetic field with allowance for radiative field effects and assumes usage of the Dirac-Schwinger equation for the neutrino wave function

$$\{i\partial_\mu\gamma^\mu - m_i\}\Psi_{\nu_i}(x) = \int M_{\nu_i}(x, x'; B)\Psi_{\nu_i}(x')dx', \quad (6.83)$$

where m_i is the mass of the neutrino field Ψ_{ν_i} and $M_{\nu_i}(x, x'; B)$ is sum of contributions to the neutrino mass operator in presence of the magnetic field. The diagonal matrix element calculated on mass shell $p_{\nu_i}^2 = m_i^2$ between the neutrino vacuum states gives the radiative correction to the mass of the neutrino in the external field

$$\Delta m_i = (p_0^{\nu_i}/m_i) \Delta p_0^{\nu_i}, \quad (6.84)$$

where the shift of the neutrino energy due to the external field presence is determined by

$$\Delta p_0^{\nu_i}(B) = \int dx dx' \bar{\nu}_i(x) M_{\nu_i}(x, x'; B) \nu_i(x'), \quad (6.85)$$

where $\nu_i(x) = (2p_{\nu_i}^0)^{-1/2} u(p) e^{-ipx}$ is the neutrino wavefunction in vacuum, p^μ is the neutrino momentum. The radiative correction to the neutrino mass (6.84) in a constant electromagnetic field given by the tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ holds the Lorentz invariant $s^\mu \tilde{F}_{\mu\nu} p^\nu$ that is connected to the neutrino polarization vector (see, for instance, in Akhiezer and Berestetsky (1965))

$$s^\mu = \left(\frac{\vec{\zeta} \cdot \vec{p}}{m_i}, \vec{\zeta} + \frac{\vec{p}(\vec{\zeta} \cdot \vec{p})}{m_i(p_0^{\nu_i} + m_i)} \right), \quad (6.86)$$

where $\vec{\zeta}$ is the neutrino rest frame spin vector overage value doubled and $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$. Note that the Lorentz invariant term $s^\mu \tilde{F}_{\mu\nu} p^\nu$ is the mean value of the spin matrix $\frac{1}{2} \sigma_{\mu\nu} F^{\mu\nu}$ in the neutrino state $u(p)$.

The proportional to the Lorentz invariant $s^\mu \tilde{F}_{\mu\nu} p^\nu$ contribution to the radiative correction to the mass (6.84) is due to the neutrino anomalous magnetic moment interaction with the external field. Following Ritus (1972), for the real part of Δm_i one gets

$$\text{Re } \Delta m_i = \mu_{\nu_i}(s^\mu \tilde{F}_{\mu\nu} p^\nu)/m_i \quad (6.87)$$

that in the neutrino rest frame transforms to

$$\text{Re } \Delta m_i = -\mu_{\nu_i}(\vec{B} \cdot \vec{\zeta}). \quad (6.88)$$

Using (6.87) it is possible to extract in the radiative correction to the mass (6.84) the term that accounts for the neutrino anomalous magnetic moment interaction with the external electromagnetic field. It can be seen that, in general, the neutrino magnetic moment depends on the strength of the external field. In particular, in a strong magnetic field the magnetic moment exhibits nonlinear dependence $\mu_{\nu_i} = \mu_{\nu_i}(B)$ on the field strength.

The ultimate goal of our present discussion is to elucidate the possible effect of a strong magnetic field on the neutrino magnetic moment in vacuum (in the absence of media). The most consistent approach is to consider in the minimal extension of the Standard Model the virtual loop processes $\nu_i \rightarrow e^- W^+ \rightarrow \nu_i$, $\nu_i \rightarrow \mu^- W^+ \rightarrow \nu_i$ and $\nu_i \rightarrow \tau^- W^+ \rightarrow \nu_i$ that contribute to $M_{\nu_i}(x, x'; B)$,

$$\begin{aligned} M_{\nu_i}(x, x'; B) = & -i \frac{g^2}{8} \sum_{l=e, \mu, \tau} |U_{li}|^2 (1 - \gamma^5) \gamma_\mu S_l(x, x'; B) \\ & \times \gamma_\nu (1 + \gamma^5) D_W^{\mu\nu}(x, x'; B), \end{aligned} \quad (6.89)$$

where U_{li} are the elements of neutrino mixing matrix, $S_l(x, x'; B)$ and $D_W^{\mu\nu}(x, x'; B)$ are the charged leptons and

¹⁰ Magnetic field dependence of the neutrino magnetic moment has been also studied by Masood *et al.* (2002).

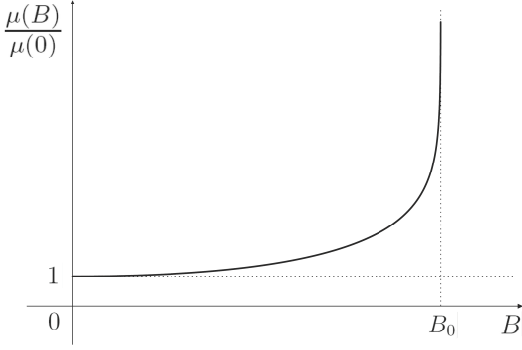


FIG. 14 Dependence of the neutrino magnetic moment on the external magnetic field strengths.

W boson propagators in the presence of external magnetic field and g is the coupling constant related to the Fermi constant, $G_F = \sqrt{2}g^2/8m_W^2$. The straightforward evaluations neglecting terms proportional to $(m_i/m_l)^2$ based on the calculation performed by Borisov *et al.* (1985) lead to the following expression for the neutrino magnetic moment in presence of weak magnetic field $B \ll B_0^e = m_e^2/e \approx 4.41 \times 10^{13}$ G,

$$\mu_{\nu_i}(B) = \mu_{\nu_i}(0) \left[1 + \frac{4}{9} \left(\frac{B}{B_0^W} \right)^2 \sum_{l=e,\mu,\tau} |U_{li}|^2 \ln \Lambda_l \right], \quad (6.90)$$

where $\Lambda_l = m_W^2/m_l^2$ and $B_0^W = m_W^2/e \sim 10^{24}$ G. In the opposite case of superstrong magnetic field $B \gg B_0^e - B_0^W$, i.e. when the magnetic field strength approaches B_0^W ,

$$\mu_{\nu_i}(B) = \frac{2}{3} \mu_{\nu_i}(0) \ln \left(\frac{B_0^W}{B_0^W - B} \right) \sum_{l=e,\mu,\tau} |U_{li}|^2 \Lambda_l, \quad (6.91)$$

where $\mu_{\nu_i}(0) = eG_F m_i / (8\sqrt{2}\pi^2)$ is the neutrino magnetic moment in vacuum (in the absence of the magnetic field).

The behavior of the neutrino magnetic moment on the magnetic field strength is shown in Fig. 14. It can be seen from (6.90) that the value $\mu_{\nu_i}(B)$ in weak magnetic fields is slightly increasing with B increase over the vacuum value $\mu_{\nu_i}(0)$. As it was already mentioned in Section IV.A to the leading order the neutrino diagonal magnetic moment is independent on mixing and also on charged leptons masses.

In case of superstrong magnetic fields, $\mu_{\nu_i}(B)$ exhibits a singular growth as $B \rightarrow B_0^W$. However, this last result should be treated with caution. As it was shown by Nielsen and Olesen (1978), Skalozub (1985, 1987) and Ambjorn and Olesen (1989), in the case when the magnetic field B is near the critical value B_0^W vacuum becomes unstable (due to the so called “zero-mode problem” in the energy-momentum relation for W boson in case $B > B_0^W$) in respect to W^+W^- pair production giving rise to W bosons condensation. Therefore, more deli-

cate treatment of the neutrino magnetic moment behavior is needed in the case of superstrong magnetic fields.

Note that the corresponding dependence of the neutrino magnetic moment on the energy of neutrino has been also discussed by Borisov *et al.* (1985). Thus, the performed studies shows the dynamical nature of the neutrino magnetic moment, in particular, its dependence on the strength of the external magnetic field and the neutrino energy. This statement can be extended to other neutrino electromagnetic characteristics.

A review on the neutrino electromagnetic properties would not be complete if the above consideration of the neutrino effective magnetic moment based on the mass operator in presence of a magnetic field were not supplemented with references to studies where discussion on the neutrino self-energy and electromagnetic vertex in the background matter and magnetic field is presented.

The neutrino self-energy and vertex function in the presence of a background media has been studied by Notzold and Raffelt (1988), Nieves and Pal (1989) and D’Olivo *et al.* (1989). The vacuum dispersion relation in the presence of a constant magnetic field has also been studied by Erdas and Feldman (1990). Finite-temperature corrections to the neutrino self-energy in the background medium has been considered by D’Olivo *et al.* (1992) and with additional account for possible presence of an external electromagnetic field by Zhukovsky *et al.* (1993), Esposito and Capone (1996) and Nieves (2003).

The general expressions for the neutrino dispersion relation in a magnetized plasma with a wide range of temperatures, chemical potentials, and magnetic field strengths has been derived by Elmfors *et al.* (1996) and Elizalde *et al.* (2002, 2004) on the basis of a neutrino self-energy calculations. The one-loop thermal self-energy of a neutrino in an arbitrary strong magnetic field has been calculated by Erdas *et al.* (1998) and Erdas and Isola (2000). The authors used the obtained results, in particular, for description of neutrino electromagnetic properties in the magnetized plasma of the early Universe.

D. Beta-decay of the neutron in a magnetic field

About forty years ago the first studies of the neutrino interactions in the presence of external electromagnetic fields were performed in by Korovina (1964) and Ternov *et al.* (1965) who considered the polarized neutron beta-decay $n \rightarrow p + e + \bar{\nu}_e$ in a magnetic field.¹¹ It was shown that the differential rate of the process exhibits the resonance spikes which appears, for the given magnetic field

¹¹ This process and the other URCA processes in Eq. (6.92), as predicted by Gamow and Schoenberg (1941), are important for energy losses in stars.

strength, each time when the final electron energy is exactly equal to one of the allowed Landau energies in the magnetic field. It was also shown that the total rate depends on the initial neutron polarization, in contrast with the field-free case when the neutron polarization dependence disappears from the rate during integration over the phase space of the process. The range of magnetic field strengths considered in these papers span up to sub-critical fields $B \geq B_0 = m^2/e = 4.41 \times 10^{13}$ G. It worth to be noted that these studies were performed before the discovery by Hewish *et al.* (1968) of pulsars, where such strong magnetic fields are believed to exist.

In the two well known papers by Matese and O'Connell (1969) and Fassio-Canuto (1969), published a few years later, the results of Korovina (1964) and Ternov *et al.* (1965) for the neutron decay rate in a magnetic field were re-derived, however there were no discussion on the asymmetry in neutrino emission.

Very strong magnetic fields are also supposed to exist in the early Universe (one of recent reviews is given by Grasso and Rubinstein (2001)). As it was discussed for the first time by Greenstein (1969) and Matese and O'Connell (1970), the weak reaction rates of the URCA processes

$$n \rightarrow p + e + \bar{\nu}_e, \quad \nu_e + n \rightleftharpoons e + p, \quad p + \bar{\nu}_e \rightleftharpoons n + e^+, \quad (6.92)$$

which determine the conversions between neutrons and protons and set the n/p - ration in various environments, can be significantly modified under the influence of magnetic fields and, as a consequence, influence the primordial nucleosynthesis affecting production of ${}^4\text{He}$. More recent studies of the magnetic field influence on processes (6.92) in connection to Big-Bang Nucleosynthesis and neutron star cooling was performed by Cheng *et al.* (1993).

The aforementioned studies of neutrino interactions in the presence of magnetic fields performed by Korovina (1964); Ternov *et al.* (1965); Matese and O'Connell (1969, 1970); Greenstein (1969); Fassio-Canuto (1969) gave the birth to *neutrino astrophysics in magnetic fields*.

The asymmetry in the neutrino emission was investigated first by Korovina (1964); Ternov *et al.* (1965) on the basis the polarized neutron beta-decay in the presence of a magnetic field rate derivation. Note that the origin of this asymmetry is the indirect influence of an external magnetic field on neutrinos through the influence on electrons and protons.

The beta-decay process can be described by the well known four-fermion Lagrangian,

$$\mathcal{L} = \frac{\tilde{G}}{\sqrt{2}} [\bar{\psi}_p \gamma_\mu (1 + g_A \gamma_5) \psi_n] [\bar{\psi}_e \gamma^\mu (1 + \gamma_5) \psi_\nu], \quad (6.93)$$

where $\tilde{G} = G_F \cos \theta_c$, θ_c is the Cabibbo angle, and $g_A \simeq 1.27$ (see Beringer *et al.* (2012)) is the axial constants.

After the standard calculations we have for the decay rate of the process

$$\Gamma = \sum_{\text{phase space}} |M|^2 \delta(E_n - E_p - E_e - E_\nu), \quad (6.94)$$

where the matrix element

$$M = \frac{\tilde{G}}{\sqrt{2}} \int d^4x [\bar{\psi}_p \gamma_\mu (1 + g_A \gamma_5) \psi_n] [\bar{\psi}_e \gamma^\mu (1 + \gamma_5) \psi_\nu] \quad (6.95)$$

accounts for the influence of the magnetic field through the wave functions of the electron and proton. For the electron wave function one has to use the exact solutions of the Dirac equation in the magnetic field given in Appendix G by Eqs. (G8), (G11), (G12) and (G13). The wave function for a proton has similar form and is given, for instance by Studenikin (1989). The initial neutron and neutrino are supposed to be not directly affected by the magnetic field and the plane waves are used for these particles wave functions.

The argument of the δ function in (6.94), being equated with zero, gives the law of energy conservation for the particles in the process, that for the case of the neutron decay at rest is

$$m_n = \sqrt{m_e^2 + 2eBN_e + p_e^2} + \sqrt{m_p^2 + 2eBN_p + p_p^2} + E_\nu, \quad (6.96)$$

where N_e and N_p are the numbers of Landau levels in the magnetic field for the electron and proton. The summation in (6.94) is performed over the phase space of the final particles: $\vec{p}_\nu, p_p^2, p_p^3, N_p, s_p, p_e^2, p_e^3, N_e, s_e$, where values $s_e, s_p = \pm 1$ denote the two possible spin states of the electron and proton. For not very strong magnetic fields $B < B_{\text{cr}} = (\Delta^2 - m_e^2)/2e = 1.8 \times 10^{14}$, where $\Delta = m_p - m_n$, the decay rate is

$$\Gamma(B) = \frac{\Gamma(0)}{2} \int \sin \theta_\nu d\theta_\nu \left\{ 1 + \frac{2(g_A^2 + g_A)}{1 + 3g_A^2} s_n \cos \theta_\nu - 4.9 \frac{eB}{\Delta^2} \left(\frac{g_A^2 - 1}{1 + 3g_A^2} \cos \theta_\nu + \frac{2(g_A^2 - g_A)}{1 + 3g_A^2} s_n \right) \right\}, \quad (6.97)$$

where θ_ν is the angle between the neutrino propagation and the magnetic field vector and $\Gamma(0)$ is the decay rate of the neutron in the absence of the magnetic field, given by

$$\Gamma(0) = \frac{\tilde{G}^2 \Delta^5}{120\pi^3} (1 + 3g_A^2) \times 0.47, \quad (6.98)$$

where $s_n = \pm 1$ correspond to the neutron spin polarization parallel or antiparallel to the magnetic field vector.

From (6.97) it is just straightforward that there is an asymmetry in the spatial distribution of neutrinos. This asymmetry is due to the parity violation in weak interactions and it is modified by the magnetic field presence.

In addition, as it is also clear from (6.97) the average momentum of antineutrinos on the magnetic field strength and the direction of propagation in respect to the magnetic field vector. That is why we consider the total effect of the antineutrino spatial distribution asymmetry as the neutrino electromagnetic properties manifestation.

Note that the same asymmetry appears in case of much stronger magnetic fields $B > B_{\text{cr}}$ as well as for other similar processes (6.92). Recently the relativistic approach to the inverse β -decay of a polarized neutron, $\nu_e + n \rightarrow p + e^-$, in a magnetic field has been developed by Shinkevich and Studenikin (2005)¹². It was shown that in strong magnetic fields the cross section can be highly anisotropic in respect to the neutrino angle. In the particular case of polarized neutrons, matter becomes even transparent for neutrinos if neutrinos propagate against the direction of neutrons polarization.

It was first claimed by Chugai (1984); Dorofeev *et al.* (1984, 1985); Zakhartsov and Loskutov (1985) that asymmetric neutrino (antineutrino) emission in the direct URCA processes (6.92) during the first seconds after a magnetized massive star collapse could provide explanations for the observed pulsar velocities. As shown by Studenikin (1988), in order to get a correct prediction for the direction and value of the kick velocity of a pulsar one has to account not only for the amount of radiated in the processes (6.92) neutrinos but also for the fact that values of the average momentum of neutrinos propagating in the opposite directions are not the same. More detailed studies of the neutrino asymmetry in relation to magnetized stars have been performed by Duan and Qian (2004); Leinson and Perez (1998); Goyal (1999); Lai and Qian (1998); Arras and Lai (1999); Gvozdev and Ognev (1999); Roulet (1998).

We recall also different other mechanisms for the asymmetry in the neutrino emission from a magnetized pulsar studied by Kusenko and Segre (1996); Bisnovatyi-Kogan (1993); Akhmedov *et al.* (1997); Lai and Qian (1998). For more complete references to the performed studies on the neutrino mechanisms of the pulsar kicks see the introductions presented by Bhattacharya and Pal (2004); Shinkevich and Studenikin (2005).

E. Neutrino pair production by an electron

It is well known that in the presence of external electromagnetic fields particles interaction processes, that are forbidden in vacuum, become possible. One may consider the corresponding processes of neutrinos interaction with real particles that could only become possible

under the influence of external electromagnetic fields as manifestation of neutrinos electromagnetic properties.

One of these processes is the production of neutrino-antineutrino pair by an electron moving in a constant magnetic field

$$e \rightarrow e + \nu_e + \bar{\nu}_e. \quad (6.99)$$

Astrophysical significance of this process, termed the synchrotron radiation of neutrinos, was discussed by Landstreet (1967). Here it worth to be noted that the possibility of $\nu\bar{\nu}$ emission by an electron through the bremsstrahlung process on a nuclei

$$e + A \rightarrow e + A + \nu_e + \bar{\nu}_e \quad (6.100)$$

was first discussed by Pontecorvo (1959) who also pointed out that for certain stages of a star evolution the proposed mechanism of $\nu\bar{\nu}$ emission might be important.

In vacuum, i.e. in the absence of the magnetic field, the process (6.99) is obviously forbidden. The dependence of the rate of the process (6.99) on the magnetic field was initially derived by Baier and Katkov (1966), Ritus (1969) and Loskutov and Zakhartsov (1969) within the local four fermion weak interaction model of Gell-Mann-Feynman. In the Weinberg-Salam model this process was considered by Ternov *et al.* (1983, 1982). In the low-energy approximation of the model for the amplitude of the process (6.99) we have used

$$M = -\frac{G_F}{\sqrt{2}} \bar{\psi}'_e \gamma_\mu (g_V + g_A \gamma_5) \psi_e \gamma^\mu \bar{\psi}_{\nu_1} (1 + \gamma_5) \psi_{\nu_2}, \quad (6.101)$$

where ψ_e and ψ'_e are the initial and final electron wave functions and ψ_{ν_1} and ψ_{ν_2} are the two neutrino wave functions. In the case of the electron $\nu\bar{\nu}$ pair emission in Eq. (6.99), $g_V = \sin^2 \theta_W + 1/2$ and $g_A = 1/2$. The effect of a constant magnetic field presence is accounted for by the wave functions of the initial and final electron that are the exact solutions of the Dirac equation in magnetic field given in Appendix G. Performing standard calculations accounting for the rotational symmetry of the problem in respect to the magnetic field \vec{B} oriented along the z axis one arrives to the rate given by Ternov *et al.* (1983, 1982)

$$\begin{aligned} \Gamma = \frac{G_F^2}{3(2\pi)^2} \sum_N \int_{|\vec{f}| \leq f_0} d^3 f [f_0^2 H_{00} \\ - (f_0^2 - |\vec{f}|^2) (H_{00} - H_{11} - H_{22} - H_{33}) \\ + |\vec{f}|^2 (H_{22} \sin^2 \theta + H_{33} \cos^2 \theta) \\ - 2f_0 |\vec{f}| (H_{20} \sin \theta + H_{30} \cos \theta) \\ + 2|\vec{f}|^2 H_{32} \cos \theta \sin \theta], \end{aligned} \quad (6.102)$$

where the sum is performed over the Landau quantum number of the final electron, $f^\mu = (f_0, \vec{f}) = p_\nu^\mu + p_{\bar{\nu}}^\mu =$

¹² This process is also important for the neutrino transport inside the magnetized pulsar and contribute to the kick velocities, as shown by Roulet (1998), Bhattacharya and Pal (2004); Duan and Qian (2004).

$p_e^\mu - p_e'^\mu$, and θ is the angle between \vec{f} and \vec{B} . The matrix elements $H_{\alpha\beta} = j_\alpha j_\beta^*$ are determined by the electron currents

$$j_\alpha = \int \bar{\psi}'_e \gamma_\mu (g_V + g_A \gamma_5) \psi_e \times \exp\{-i[(\kappa_1 + \eta_1)x + (\kappa_2 + \eta_2)y]\} dx dy \quad (6.103)$$

κ_i and η_i are the corresponding neutrino and antineutrino momenta components. The functions $H_{\alpha\beta}$ are expressed in terms of quadratic combinations of Laguerre functions which depend on the argument $\rho = |\vec{f}|^2 \sin^2 \theta / (2eB)$. In the case of the ultrarelativistic electron energies the resulting expressions for the rate depend on the electromagnetic field dynamical parameter

$$\chi = \frac{e\sqrt{(F_{\mu\nu}p^\nu)^2}}{m_e^2} = \frac{B}{B_0} \frac{p_0}{m_e}. \quad (6.104)$$

Integration in (6.102) can be performed analytically. The final expressions for the rate Γ was obtained by Ternov *et al.* (1983, 1982):

$$\Gamma = \frac{G_F^2 m_e^6 \chi^5}{1152 \sqrt{3} \pi^3 p_0} \left[49g^2 + 437g_A^2 \right], \quad (6.105)$$

for $\chi \ll 1$ and

$$\Gamma = \frac{G_F^2 m_e^6 \chi^2}{216 \pi^3 p_0} (g_V^2 + g_A^2) \left[\ln \chi - C - \frac{\ln 3}{2} - \frac{5}{6} \right], \quad (6.106)$$

for $\chi \gg 1$, where $C = 0.577$ is the Euler constant.

From Eqs. (6.105) and (6.106) one can see that rate is governed by the value of the parameter χ . It follows that the rate is significantly dependent on the magnetic field strength and the initial electron energy. Therefore, for ultrarelativistic energies and strong enough magnetic fields the $\nu\bar{\nu}$ synchrotron radiation by an electron can be important for astrophysics.

As it have been demonstrated, for instance by Kaminker *et al.* (1992), more consistent consideration of the process $e \rightarrow e + \nu_e + \bar{\nu}_e$ appropriate for astrophysical applications implies account for the presence of background matter in addition to an external magnetic field.

F. Neutrino pair production by a strong magnetic field

Over the years, starting from the observation of Klein (1929), it has been known that vacuum is not stable under the influence of an external electrical field. It was also shown long ago by Schwinger (1951) that electron-positron pairs can be produced from vacuum in presence of a strong electrical field with strength that exceeds the critical value $E_0 = m_e^2/e$. On the contrary, it is often claimed that in strong magnetic fields vacuum is stable. However, the last statement is correct in case of minimal

electromagnetic coupling. Recently nonminimal electromagnetic coupling of neutral particles with electromagnetic fields through particles magnetic moment has been considered by Lin (1999), Lee and Yoon (2006) and Lee and Yoon (2007). And it has been shown that neutral particle-antiparticle pairs can be created from vacuum through Pauli interaction of nonzero (anomalous) magnetic moment with external electromagnetic fields.

Following these lines, we would like to discuss another rather spectacular manifestation of neutrino electromagnetic properties that have been considered recently by Lee and Yoon (2008) and Lee (2011). In particular, it has been shown that in case neutrinos have non vanishing magnetic moments vacuum is unstable against neutrino-antineutrino pair production through the Pauli interaction in strong enough magnetic fields.

Consider a massive neutrino ν_i with non-zero magnetic (anomalous) moment μ_{ν_i} that have Pauli interaction with an external electromagnetic field $F_{\mu\nu}$. The corresponding Dirac equation for the neutrino wave function is

$$\{i\partial_\mu \gamma^\mu - m_{\nu_i} + \frac{\mu_{\nu_i}}{2} \sigma_{\mu\nu} F^{\mu\nu}\} \Psi_{\nu_i}(x) = 0, \quad (6.107)$$

and the neutrino energy spectrum in case of a constant magnetic field B is given, for instance, by Khalilov (2002)

$$p_{\nu_i}^0 = \pm \sqrt{p_3^2 + (\sqrt{m_{\nu_i}^2 + p_\perp^2} - \mu_{\nu_i} B s)^2}, \quad (6.108)$$

where p_3 and p_\perp are the longitudinal and transversal momentum to the magnetic field direction and $s = \pm 1$ are two spin projections along the magnetic field.

From (6.108) it is just straightforward that the energy gap between the positive and the negative energy states disappears when the magnetic field strength B reaches the critical value $B_{cr} = m_{\nu_i}/\mu_{\nu_i}$. This can be considered as indication for instability of vacuum in respect to the electromagnetic production of the neutrino-antineutrino pairs through Pauli interaction in a strong magnetic field.

The most prominent environment for the discussed mechanism of neutrino-antineutrino pairs production is the vicinity of astrophysical compact objects where, following to Duncan and Thompson (1992) and Harding (2006), very strong magnetic fields of order $B \sim 10^{15} \text{G}$ and even stronger may exist. However, an estimation of the critical magnetic field $B_{cr} = m_{\nu_i}/\mu_{\nu_i}$, taking the magnetic moment as large as the experimental upper bound $\mu_\nu = 10^{-11} \mu_0$ and the neutrino mass to be $m_{\nu_i} = 10^{-2} \text{eV}$, gives $B_{cr} \sim 10^{17} \text{G}$ that may not provide conditions for sufficient neutrino-antineutrino pair production through Pauli interaction.

Note that the effect discussed is possible in theories beyond the Standard Model. In the lowest order of the Standard Model, $\nu\bar{\nu}$ pair production by the charged particles annihilation is through Z^0 channel, whereas in case neutrinos have non-zero magnetic moment they can be also produced through the photon channel, as it was

shown by Barut *et al.* (1982), Sehgal and Weber (1992) and Deshpande and Sharma (1991).

It has been noted by Lee (2011) that the cross section of pair production becomes dominated by Pauli interaction over the process through Z^0 channel as the energy is increasing. Thus, substantial increase of neutrino-antineutrino pair production rates are expected at the LHC energies ($E_{\text{LHC}}^{\text{CM}} > 10 \text{ TeV}$) and ultra high energy cosmic ray experiments ($E_{\text{UHECR}}^{\text{CM}} \sim 100 \text{ TeV}$) if the neutrino magnetic moments are not much smaller than $10^{-10} \mu_B$.

An interesting possibility of Majorana neutrinos with transition magnetic moment production through photon channel with Pauli interaction at the annihilation of charged particles in colliding experiments have been also considered in by Lee (2011).

G. Energy quantization in rotating media

In Section VII.A we will discuss the possibility of non-zero neutrino electric charge, that is predicted in a set of Standard Model extensions. If a neutrino is really a millicharged particle, in the presence of a constant magnetic field it behaves in a way similar to an electron. In particular, the energy of a millicharged neutrino is quantized in a magnetic field (see Appendix G)

$$p_0^\nu = \sqrt{m_\nu^2 + p_3^2 + 2q_\nu B N_\nu}, \quad (6.109)$$

where q_ν is millicharge of the neutrino and $N_\nu = 0, 1, 2, \dots$ is the Landau number of the millicharged neutrino energy levels. The corresponding radius of the neutrino classical orbits in the magnetic field is given by Balantsev *et al.* (2011)

$$\langle R_B^\nu \rangle = \sqrt{\frac{2N_\nu}{q_\nu B}}. \quad (6.110)$$

It is interesting to compare the radius of classical orbits in a magnetic field of the millicharged neutrino, $\langle R_B^\nu \rangle$, with that of the electron, $\langle R_B^e \rangle$. If the relativistic electron and millicharged neutrino are moving with the same energy in a constant magnetic field then the ratio of orbits radiuses is equal to the inverse ratio of electric charges

$$\frac{\langle R_B^\nu \rangle}{\langle R_B^e \rangle} = \frac{e}{q_\nu}, \quad (6.111)$$

if for both particles the momentum components along the magnetic field vector are zero. From the obtained estimation for the ratio of orbits radiuses, taking into account existing experimental constraints on neutrino millicharge, we conclude that for the same strength of the external magnetic field the motion of a charged neutrino is much less localized as compared with an electron motion.

The same method of wave equations exact solutions that is used in studies of charged particles under the influence of external electromagnetic fields (including millicharged neutrinos and neutrinos with non zero magnetic moment, see above discussions of this Section and Appendix G), as it has been explicitly demonstrated by Studenikin and Ternov (2005) and Studenikin (2008), can be also used for investigations of neutrinos moving in the background matter. In particular, using the method of exact solutions for a neutrino wave function in the presence of matter it has been shown by Grigoriev *et al.* (2007) and Studenikin (2008) that the energy spectrum of a neutrino moving in a rotating media is quantized. This effect is very similar to charged particles energy quantization in a magnetic field.

The neutrino wave function exactly accounting for the neutrino interaction with matter can be obtained by solving the modified Dirac equation given by Studenikin and Ternov (2005) (see Appendix G),

$$\left\{ i\gamma_\mu \partial^\mu - \frac{1}{2}\gamma_\mu(1 + \gamma_5)f^\mu - m_\nu \right\} \Psi(x) = 0. \quad (6.112)$$

In case an electron neutrino is propagating through a rotating matter composed of neutrons then the matter potential, according to Balantsev *et al.* (2009, 2011), is

$$f^\mu = -G(n, n\vec{v}), \quad \vec{v} = (-\omega y, \omega x, 0), \quad (6.113)$$

where ω is the angular frequency of matter rotation around the z axis and $G = G_F/\sqrt{2}$. The neutrino energy spectrum obtained by Balantsev *et al.* (2009, 2011)

$$p_0 = \sqrt{m_\nu^2 + p_3^2 + p_\perp^2} - Gn, \quad (6.114)$$

contains the transversal momentum

$$p_\perp = 2\sqrt{NG\omega}, \quad N = 0, 1, 2, \dots \quad (6.115)$$

that is quantized (see also Grigoriev *et al.* (2007)). The quantum number N determines also the radius of classical orbits of neutrino in rotating matter (it is supposed that $N \gg 1$ and $p_3 = 0$),

$$R = \sqrt{\frac{N}{Gn\omega}}. \quad (6.116)$$

It is shown by Studenikin (2008) that for low energy neutrinos it can be $R \sim R_{NS} = 10 \text{ km}$ that might be thought to be of interest in applications for neutron stars.

It is interesting to note that within the quasiclassical approach the neutrino binding on circular orbits is due to an effective force that is orthogonal to the particle speed. And an analogy between a charged particle motion in a magnetic field and a neutrino motion in a rotating matter can be established (Studenikin (2008)). It is possible to explain the neutrino quasiclassical circular orbits as a result of action of the attractive central force,

$$\vec{F}_m^{(\nu)} = q_m^{(\nu)} \vec{\beta} \times \vec{B}_m, \quad \vec{B}_m = \vec{\nabla} \times \vec{A}_m, \quad \vec{A}_m = n\vec{v}, \quad (6.117)$$

where the effective neutrino “charge” in matter (composed of neutrons in the discussed case) is $q_m^{(\nu)} = -G$, whereas \vec{B}_m and \vec{A}_m play the roles of effective “magnetic” field and the correspondent “vector potential”. Like the magnetic part of the Lorentz force, $\vec{F}_m^{(\nu)}$ is orthogonal to the speed $\vec{\beta}$ of the neutrino.

For the most general case the “matter induced Lorentz force” is given by

$$\vec{F}_m^{(\nu)} = q_m^{(\nu)} \vec{E}_m + q_m^{(\nu)} \vec{\beta} \times \vec{B}_m, \quad (6.118)$$

where the effective “electric” and “magnetic” fields are respectively,

$$\vec{E}_m = -\vec{\nabla}n - \vec{v} \frac{\partial n}{\partial t} - n \frac{\partial \vec{v}}{\partial t}, \quad (6.119)$$

and

$$\vec{B}_m = n \vec{\nabla} \times \vec{v} - \vec{v} \times \vec{\nabla}n. \quad (6.120)$$

The force acting on a neutrino, produced by the first term of the effective “electric” field in the neutron matter, was considered also by Loeb (1990) and the quasiclassical treatment of a neutrino motion in the electron plasma was considered by Mendonca *et al.* (1998).

Note that while considering a neutrino effective electromagnetic interactions with media an effective electric charge of the neutrino has been introduced by Oraevsky *et al.* (1986); Oraevsky and Semikoz (1987); Oraevsky *et al.* (1994); Nieves and Pal (1994); Mendonca *et al.* (1998); Bhattacharya *et al.* (2002); Nieves (2003); Studenikin (2008).

In the most general case the description of the millicharged neutrino with anomalous magnetic moment motion in the presence of matter and external electromagnetic fields can be obtained by solving the modified Dirac equation

$$\left\{ \gamma_\mu (p^\mu + q_\nu A^\mu) - \frac{1}{2} \gamma_\mu (1 + \gamma_5) f^\mu - \frac{i}{2} \mu_\nu \sigma_{\mu\nu} F^{\mu\nu} - m_\nu \right\} \Psi(x) = 0, \quad (6.121)$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and A^μ is the electromagnetic field potential, μ_ν is the neutrino anomalous magnetic moment. For several particular cases this equation can be solved exactly and the neutrino wave functions and corresponding energy spectra can be found (Studenikin and Tokarev, 2012; Balantsev *et al.*, 2012, 2013). In particular, for a neutrino moving in a rotating matter with the potential

$$f^\mu = -GN_n(1, -\epsilon y\omega, \epsilon x\omega, 0). \quad (6.122)$$

and superimposed constant electric \vec{E} and magnetic field \vec{B} , $\epsilon = \pm 1$ corresponds to parallel and antiparallel directions of vectors $\vec{\omega}$ and \vec{B} , for the neutrino energy spectrum we obtain

$$p_0 = \sqrt{p_3^2 + 2N|2GN_n\omega - \epsilon q_\nu B| + m_\nu^2} - GN_n - q_\nu \phi, \quad (6.123)$$

where ϕ is the scalar potential of the electric field. In this case the generalized effective Lorentz force is

$$\vec{F}_{eff} = q_{eff} \vec{E}_{eff} + q_{eff} [\vec{\beta} \times \vec{B}_{eff}]. \quad (6.124)$$

Here $\vec{\beta}$ is the neutrino speed and

$$\begin{aligned} q_{eff} \vec{E}_{eff} &= q_m \vec{E}_m + q_\nu \vec{E}, \\ q_{eff} \vec{B}_{eff} &= |q_m B_m + \epsilon q_\nu B| \vec{e}_z, \end{aligned} \quad (6.125)$$

where $q_m, \vec{B}_m, \vec{E}_m$ are the matter induced “charge”, “electric” and “magnetic” fields correspondingly,

$$q_m = -G, \quad \vec{E}_m = -\vec{\nabla}N_n, \quad \vec{B}_m = -2N_n \vec{\omega}. \quad (6.126)$$

Note that the effective Lorentz force (6.124), that directly follows from the exact form of the obtained energy spectrum (6.123), is generated by both weak and electromagnetic interactions. The effect of the millicharged neutrino energy quantization in a rotating magnetized matter is discussed in Studenikin and Tokarev (2012, 2013) where is shown that the neutrino trapping in circular orbits exist due to the neutrino millicharge interaction with the magnetic field and also due to neutrino weak interaction with the rotating matter.

Under the influence of the effective Lorentz force (6.125) the neutrino will move with acceleration given by

$$\vec{a} = \frac{1}{m_\nu} \left(G \vec{\nabla}N_n + q_\nu \vec{\nabla}\phi + |2GN_n\omega - \epsilon q_\nu B| \vec{\beta} \times \vec{e}_z \right), \quad (6.127)$$

where \vec{e}_z is a unit vector in the direction of the magnetic field and matter rotation. The accelerated neutrino should produce the electromagnetic radiation. In the quasiclassical treatment the radiation power of induced electromagnetic radiation is given by

$$I_{LC\nu} = \frac{2q_\nu^2}{3} \left(\frac{\vec{a}^2}{(1 - |\vec{\beta}|^2)^2} + \frac{(\vec{a} \cdot \vec{\beta})^2}{(1 - |\vec{\beta}|^2)^3} \right). \quad (6.128)$$

Such a mechanism of the neutrino electromagnetic radiation due to the neutrino millicharge, that can be emitted in the presence of the nonuniform rotating matter and electromagnetic fields, is termed in Studenikin and Tokarev (2012) the “Light of (milli)Charged Neutrino” ($LC\nu$). It should be stressed, that the phenomenon exist even in the absence of the electromagnetic fields, when the acceleration (6.127) is produced only due to the weak interactions of neutrinos with the background particles. So that the discussed mechanism is of different nature that one of the cyclotron radiation of a charged particle in magnetic fields.

The $LC\nu$ mechanism would manifest itself during the neutrino propagation from the central part of a rotating neutron star outwards through the crust. The gradient of

the matter density (the density variation along the neutrino path) gives the following contribution to the $LC\nu$ radiation power (see Eq. (6.127))

$$I_{LC\nu} = \frac{2q_\nu^2}{3m_\nu^2} \left(G\vec{\nabla} N_n \right)^2, \quad (6.129)$$

and the effect of the matter rotation yields

$$I_{LC\nu} = \frac{2q_\nu^2\gamma^2}{3m_\nu^2} (-\epsilon q_\nu B + 2GN_n\omega)^2, \quad (6.130)$$

where $\gamma = (1 - |\vec{\beta}|^2)^{-1/2}$. The numerical estimations, that account for the $LC\nu$ power for the present limits on the neutrino millicharge and for a realistic gradient of a neutron star matter density $|G\vec{\nabla} N_n| \sim 1\text{eV}/1\text{km}$ and the rotation frequency $\omega \sim 2\pi \times 10^3 \text{ s}^{-1}$, show that the role of the $LC\nu$ in the explosion energetics is negligible in respect to the total energy of the collapse. However, as it is discussed in Section VII (Oraevsky *et al.*, 1994; Nieves, 2003; Duan and Qian, 2004), in the presence of a dense plasma the induced neutrino effective electric charge can be reasonably large. In addition, the phenomenon is of interest for astrophysics in light of the recently reported hints of ultra-high energy PeV neutrinos observed by Ice-Cube (Aartsen *et al.*, 2013).

VII. CHARGE AND ANAPOLE FORM FACTORS

The magnetic and electric dipole moments are the most studied electromagnetic properties in theoretical and experimental works, but some attention has been also devoted to the possibility that neutrinos have very small electric charges, usually called “millicharges”. Moreover, even if neutrinos are exactly neutral, they can have non-zero charge radii, which can be probed in scattering experiments. In Subsections VII.A and VII.B we review the theory of electric charge and charge radius, respectively, and we present the corresponding experimental limits. In Subsection VII.C we discuss the neutrino anapole moment, which is the most mysterious neutrino electromagnetic property.

A. Neutrino electric charge

It is usually believed (Bernstein *et al.*, 1963) that the neutrino electric charge is zero. This is often thought to be attributed to gauge invariance and anomaly cancellation constraints imposed in the Standard Model. In the Standard Model of $SU(2)_L \times U(1)_Y$ electroweak interactions it is possible to get a general proof that neutrinos are electrically neutral (Babu and Mohapatra, 1990b; Foot *et al.*, 1990, 1993). The electric charges of particles in this model are related to the $SU(2)_L$ and $U(1)_Y$ eigen-

values by (see Table I)

$$Q = I_3 + \frac{Y}{2}. \quad (7.1)$$

In the Standard Model without right-handed neutrinos ν_R the triangle anomalies cancellation constraints (the requirement of renormalizability) lead to certain relations among particles hypercharges Y , that are enough to fix all Y , so that hypercharges, and consequently electric charges, are quantized (Foot *et al.*, 1990, 1993). In this case, neutrinos are electrically neutral.

The direct calculation of the neutrino charge in the Standard Model under the assumption of a vanishing neutrino mass in different gauges and with use of different methods is presented in Bardeen *et al.* (1972); Beg *et al.* (1978); Marciano and Sirlin (1980); Sakakibara (1981); Lucio *et al.* (1984, 1985); Cabral-Rosetti *et al.* (2000). For the flavor massive Dirac neutrino the one-loop contributions to the charge, in the context of the minimal extension of the Standard Model within the general R_ξ gauge, were considered in Dvornikov and Studenikin (2004a,b). By these direct calculations within the mentioned above theoretical frameworks it is proven that at least at one-loop level approximation neutrino electric charge is gauge independent and vanish.

However, if the neutrino has a mass, the statement that a neutrino electric charge is zero is not so evident as it meets the eye. It is not entirely assured that the electric charge should be quantized (see Raffelt (1996, 1999a)). We recall here that the problem of charge quantization has been always a mystery within quantum electrodynamics (Davidson *et al.*, 1991). The absence of an algebraic quantization of the charge eigenvalues in electrodynamics led to the proposal (Dirac, 1931) of a possible topological explanation leading to magnetic monopoles.

The strict requirements for charge quantization may also disappear in extensions of the standard $SU(2)_L \times U(1)_Y$ electroweak interaction models if right-handed neutrinos ν_R with $Y \neq 0$ are included. In this case the uniqueness of particles hypercharges Y is lost (hypercharges are no more fixed) and in the absence of hypercharge quantization the electric charge gets “dequantized” (Foot *et al.*, 1990, 1993). As a result, neutrinos may become electrically millicharged particles.

In general, the situation with charge quantization is different for Dirac and Majorana neutrinos. As it was shown by Babu and Mohapatra (1990b), charge dequantization for Dirac neutrinos occurs in the extended Standard Model with right-handed neutrinos ν_R and also in a wide class of models that contain an explicit $U(1)$ symmetry. On the contrary, if the neutrino is a Majorana particle, the arbitrariness of hypercharges in this kind of models is lost, leading to electric charge quantization and hence to neutrino neutrality (Babu and Mohapatra, 1990b).

Finally, while there are other Standard Model extensions (superstrings, GUTs etc) that provide enforcing of charge quantization, there are also models (for instance, with a “mirror sector” (Holdom, 1986)) that predict the existence of new particles of arbitrary mass and small (unquantized) electric charge, in which neutrino can be a millicharged particle.

Some approximate constraints obtained with various assumptions from reactor, accelerator and astrophysical data are listed in Tab. IV (see also Beringer *et al.* (2012)).

The most severe experimental constraint on neutrino electric charges is that on the effective electron neutrino charge q_{ν_e} , which can be obtained from electric charge conservation in neutron beta decay $n \rightarrow p + e^- + \bar{\nu}_e$, from the experimental limits on the neutrality of matter which constrain the sum of the proton and electron charges, $q_p + q_e$, and from the experimental limits on the neutron charge q_n (Raffelt, 1996, 1999a). Several experiments which measured the neutrality of matter give their results in terms of

$$q_{\text{mat}} = \frac{Z(q_p + q_e) + Nq_n}{A}, \quad (7.2)$$

where $A = Z + N$ is the atomic mass of the substance under study, Z is its atomic number and N is its neutron number. From electric charge conservation in neutron beta decay, we have

$$q_{\nu_e} = q_n - (q_p + q_e) = \frac{A}{Z} (q_n - q_{\text{mat}}). \quad (7.3)$$

The best recent bound on the neutrality of matter (Bressi *et al.*, 2011),

$$q_{\text{mat}} = (-0.1 \pm 1.1) \times 10^{-21} e, \quad (7.4)$$

has been obtained with SF_6 , which has $A = 146.06$ and $Z = 70$. Using the independent measurement of the charge of the free neutron (Baumann *et al.*, 1988)

$$q_n = (-0.4 \pm 1.1) \times 10^{-21} e, \quad (7.5)$$

we obtain

$$q_{\nu_e} = (-0.6 \pm 3.2) \times 10^{-21} e. \quad (7.6)$$

This value is compatible with the neutrality of matter limit in Tab. IV, which has been derived (Raffelt, 1996, 1999a) from the value of q_n in Eq. (7.5) and $q_{\text{mat}} = (0.8 \pm 0.8) \times 10^{-21} e$ (Marinelli and Morpurgo, 1984).

It is also interesting that the effective charge of $\bar{\nu}_e$ can be constrained by the SN 1987A neutrino measurements taking into account that galactic and extragalactic magnetic field can lengthen the path of millicharged neutrinos and requiring that neutrinos with different energies arrive on Earth within the observed time interval of a few seconds (Barbiellini and Cocconi, 1987):

$$|q_{\nu_e}| \lesssim 3.8 \times 10^{-12} \frac{(E_\nu/10 \text{ MeV})}{(d/10 \text{ kpc})(B/1 \mu\text{G})} \sqrt{\frac{\Delta t/t}{\Delta E_\nu/E_\nu}}, \quad (7.7)$$

considering a magnetic field B acting over a distance d and the corresponding time $t = d/c$. $E_\nu \approx 15 \text{ MeV}$ is the average neutrino energy, $\Delta E_\nu \approx E_\nu/2$ is the energy spread, and $\Delta t \approx 5 \text{ s}$ is the arrival time interval. Barbiellini and Cocconi (1987) considered 2 cases:

1. An intergalactic field $B \approx 10^{-3} \mu\text{G}$ acting over the whole path $d \simeq 50 \text{ kpc}$, which corresponds to $t \simeq 5 \times 10^{12} \text{ s}$, gives

$$|q_{\nu_e}| \lesssim 2 \times 10^{-15} e. \quad (7.8)$$

2. An galactic field $B \approx 1 \mu\text{G}$ acting over a distance $d \simeq 10 \text{ kpc}$, which corresponds to $t \simeq 1 \times 10^{12} \text{ s}$, gives

$$|q_{\nu_e}| \lesssim 2 \times 10^{-17} e. \quad (7.9)$$

The last two limits in Tab. IV have been obtained (Gninenko *et al.*, 2007; Studenikin, 2013) considering the results of reactor neutrino magnetic moment experiments (see Sections IV.C and IV.E). The differential cross section of the $\bar{\nu}_e e^-$ elastic scattering process due to an effective neutrino charge q_{ν_e} is given by (see Berestetskii *et al.* (1979))

$$\left(\frac{d\sigma}{dT_e} \right)_{\text{charge}} \simeq 2\pi\alpha \frac{1}{m_e T_e^2} q_{\nu_e}^2. \quad (7.10)$$

In reactor experiments the neutrino magnetic moment is searched by considering data with $T_e \ll E_\nu$, for which the ratio of the charge cross section (7.10) and the magnetic moment cross section in Eq. (4.29), for which we consider only the dominant part proportional to $1/T_e$, is given by

$$R = \frac{(d\sigma/dT_e)_{\text{charge}}}{(d\sigma/dT_e)_{\text{mag}}} \simeq \frac{2m_e}{T_e} \frac{(q_{\nu_e}/e)^2}{(\mu_{\nu_e}/\mu_B)^2} \quad (7.11)$$

Considering an experiment which does not observe any effect of μ_{ν_e} and obtains a limit on μ_{ν_e} , it is possible to obtain a bound on q_{ν_e} by demanding that the effect of q_{ν_e} is smaller than that of μ_{ν_e} , i.e that $R \lesssim 1$:

$$q_{\nu_e}^2 \lesssim \frac{T_e}{2m_e} \left(\frac{\mu_{\nu_e}}{\mu_B} \right)^2 e. \quad (7.12)$$

The last limit in Tab. IV has been obtained from the 2012 results (Beda *et al.*, 2012) of the GEMMA experiment, considering T_e at the experimental threshold of 2.8 keV .

Let us finally note that a strong limit on a generic neutrino electric charge q_ν can be obtained by considering the influence of millicharged neutrinos on the rotation a magnetized star which is undergoing a supernova explosion (the Neutrino Star Turning mechanism, νST) (Studenikin and Tokarev, 2012). During the supernova explosion, the escaping millicharged neutrinos move along curved orbits inside the rotating magnetized

Limit	Method	Reference
$ \mathfrak{q}_{\nu_\tau} \lesssim 3 \times 10^{-4} e$	SLAC e^- beam dump	Davidson <i>et al.</i> (1991)
$ \mathfrak{q}_{\nu_\tau} \lesssim 4 \times 10^{-4} e$	BEBC beam dump	Babu <i>et al.</i> (1994)
$ \mathfrak{q}_\nu \lesssim 6 \times 10^{-14} e$	Solar cooling (plasmon decay)	Raffelt (1999a)
$ \mathfrak{q}_\nu \lesssim 2 \times 10^{-14} e$	Red giant cooling (plasmon decay)	Raffelt (1999a)
$ \mathfrak{q}_{\nu_e} \lesssim 3 \times 10^{-21} e$	Neutrality of matter	Raffelt (1999a)
$ \mathfrak{q}_{\nu_e} \lesssim 3.7 \times 10^{-12} e$	Nuclear reactor	Gninenko <i>et al.</i> (2007)
$ \mathfrak{q}_{\nu_e} \lesssim 1.5 \times 10^{-12} e$	Nuclear reactor	Studenikin (2013)

TABLE IV Approximate limits for different effective neutrino charges. The limits on \mathfrak{q}_ν apply to all flavors.

star and slow down the rotation of the star. This mechanism could prevent the generation of a rapidly rotating pulsar in the supernova explosion. Imposing that the frequency shift of a forming pulsar due to the νST mechanism is less than a typical observed frequency of 0.1 s^{-1} and assuming a magnetic field of the order of 10^{14} Gauss, Studenikin and Tokarev (2012) obtained

$$|\mathfrak{q}_\nu| \lesssim 1.3 \times 10^{-19} e. \quad (7.13)$$

Note that this limit is much stronger than the astrophysical limits in Tab. IV.

B. Neutrino charge radius

Even if the electric charge of a neutrino is zero, the electric form factor $\mathfrak{f}_Q(q^2)$ can contain nontrivial information about the neutrino electric properties. In fact, a neutral particle can be characterized by a (real or virtual) superposition of two different charge distributions of opposite signs, which is described by a form factor $\mathfrak{f}_Q(q^2)$ which is non-zero for $q^2 \neq 0$.

The neutrino charge radius is determined by the second term in the expansion of the neutrino charge form factor $\mathfrak{f}_Q(q^2)$ in series of powers of q^2 :

$$\mathfrak{f}_Q(q^2) = \mathfrak{f}_Q(0) + q^2 \left. \frac{d\mathfrak{f}_Q(q^2)}{dq^2} \right|_{q^2=0} + \dots \quad (7.14)$$

In the so-called “Breit frame”, in which $q_0 = 0$, the charge form factor $\mathfrak{f}_Q(q^2)$ depends only on $|\vec{q}| = \sqrt{-q^2}$ and can be interpreted as the Fourier transform of a spherically symmetric charge distribution $\rho(r)$, with $r = |\vec{x}|$:

$$\mathfrak{f}_Q(q^2) = \int \rho(r) e^{-i\vec{q} \cdot \vec{x}} d^3x = \int \rho(r) \frac{\sin(|\vec{q}|r)}{|\vec{q}|r} d^3x. \quad (7.15)$$

Deriving with respect to $q^2 = -|\vec{q}|^2$, we obtain

$$\frac{d\mathfrak{f}_Q(q^2)}{dq^2} = \int \rho(r) \frac{\sin(|\vec{q}|r) - |\vec{q}|r \cos(|\vec{q}|r)}{2q^3 r} d^3x, \quad (7.16)$$

and

$$\lim_{q^2 \rightarrow 0} \frac{d\mathfrak{f}_Q(q^2)}{dq^2} = \int \rho(r) \frac{r^2}{6} d^3x = \frac{\langle r^2 \rangle}{6}. \quad (7.17)$$

Therefore, the squared neutrino charge radius is given by

$$\langle r^2 \rangle = 6 \left. \frac{d\mathfrak{f}_Q(q^2)}{dq^2} \right|_{q^2=0}. \quad (7.18)$$

Note that $\langle r^2 \rangle$ can be negative, because the charge density $\rho(r)$ is not a positively defined quantity.

The theory of the neutrino charge radius has a long history, with some controversies which are shortly summarized in the following.

In one of the first studies (Bardeen *et al.*, 1972), it was claimed that in the Standard Model and in the unitary gauge the neutrino charge radius is ultraviolet-divergent and so it is not a physical quantity. A direct one-loop calculation (Dvornikov and Studenikin, 2004a,b) of proper vertices (Fig. 6) and $\gamma - Z$ self-energy (Fig. 7) contributions to the neutrino charge radius performed in a general R_ξ gauge for a massive Dirac neutrino gave also a divergent result. However, it was shown (Lee, 1972), using the unitary gauge, that by including in addition to the usual terms also contributions from diagrams of the neutrino-lepton neutral current scattering (Z boson diagrams), it is possible to obtain for the neutrino charge radius a gauge-dependent but finite quantity. Later on, it was also shown (Lee and Shrock, 1977) that in order to define the neutrino charge radius as a physical quantity one has to consider additional box diagrams and that in combination with contributions from the proper diagrams it is possible to obtain a finite and gauge-independent value for the neutrino charge radius. In this way, the neutrino electroweak radius was defined by Lucio *et al.* (1984, 1985) and an additional set of diagrams that give contribution to its value was discussed by Degraasi *et al.* (1989). Finally, in a series of papers by Bernabeu *et al.* (2000, 2002, 2004) the neutrino electroweak radius as a physical observable has been introduced. In the corresponding calculations, performed in the one-loop approximation including additional terms from the $\gamma - Z$ boson mixing and the box diagrams involving W and Z bosons, the following gauge-invariant result for the neutrino charge radius have been obtained:

$$\langle r_{\nu_\ell}^2 \rangle_{\text{SM}} = \frac{G_F}{4\sqrt{2}\pi^2} \left[3 - 2 \log \left(\frac{m_\ell^2}{m_W^2} \right) \right], \quad (7.19)$$

Method	Experiment	Limit [cm ²]	CL	Reference
Reactor $\bar{\nu}_e e^-$	Krasnoyarsk	$ \langle r_{\nu_e}^2 \rangle < 7.3 \times 10^{-32}$	90%	Vidyakin <i>et al.</i> (1992)
	TEXONO	$-4.2 \times 10^{-32} < \langle r_{\nu_e}^2 \rangle < 6.6 \times 10^{-32}$	90%	Deniz <i>et al.</i> (2010) ^a
Accelerator $\nu_e e^-$	LAMPF	$-7.12 \times 10^{-32} < \langle r_{\nu_e}^2 \rangle < 10.88 \times 10^{-32}$	90%	Allen <i>et al.</i> (1993) ^a
	LSND	$-5.94 \times 10^{-32} < \langle r_{\nu_e}^2 \rangle < 8.28 \times 10^{-32}$	90%	Auerbach <i>et al.</i> (2001) ^a
Accelerator $\nu_\mu e^-$	BNL-E734	$-4.22 \times 10^{-32} < \langle r_{\nu_\mu}^2 \rangle < 0.48 \times 10^{-32}$	90%	Ahrens <i>et al.</i> (1990) ^a
	CHARM-II	$ \langle r_{\nu_\mu}^2 \rangle < 1.2 \times 10^{-32}$	90%	Vilain <i>et al.</i> (1995) ^a

^a The published limits are half, because they use a convention which differs by a factor of 2 (see also Hirsch *et al.* (2003)).

TABLE V Experimental limits for the electron neutrino charge radius.

where m_W and m_ℓ are the W boson and lepton masses ($\ell = e, \mu, \tau$). This result, however, revived the discussion (Fujikawa and Shrock, 2003, 2004; Papavassiliou *et al.*, 2004; Bernabeu *et al.*, 2005a) on the definition of the neutrino charge radius. Numerically, Eq. (7.19) gives (Bernabeu *et al.*, 2000, 2004)

$$\langle r_{\nu_e}^2 \rangle_{\text{SM}} = 4.1 \times 10^{-33} \text{ cm}^2, \quad (7.20)$$

$$\langle r_{\nu_\mu}^2 \rangle_{\text{SM}} = 2.4 \times 10^{-33} \text{ cm}^2, \quad (7.21)$$

$$\langle r_{\nu_\tau}^2 \rangle_{\text{SM}} = 1.5 \times 10^{-33} \text{ cm}^2. \quad (7.22)$$

These value are of the same order of magnitude of the numerical estimation $\langle r_{\nu_e}^2 \rangle \approx 10^{-33} \text{ cm}^2$ obtained by Lucio *et al.* (1985).

The effects of new physics beyond the Standard Model can contribute to the neutrino charge radius. However, Novales-Sanchez *et al.* (2008) have shown that in the context of an effective electroweak Yang-Mills theory the anomalous $WW\gamma$ vertex contribution to the neutrino effective charge radius is smaller than about 10^{-34} cm^2 , which is one order of magnitude smaller than the Standard Model values in Eqs. (7.20)–(7.22).

The neutrino charge radius has an effect in the scattering of neutrinos with charged particles. The most useful process is the elastic scattering with electrons, which has been discussed in Section IV.C in connection with the searches of neutrino magnetic moments. Since in the ultrarelativistic limit the charge form factor conserves the neutrino helicity (see Appendix C), a neutrino charge radius contributes to the weak interaction cross section $(d\sigma/dT_e)_{\text{SM}}$ of $\nu_\ell e^-$ elastic scattering through the following shift of the vector coupling constant $g_V^{\nu_\ell}$ (Grau and Grifols, 1986; Degraasi *et al.*, 1989; Vogel and Engel, 1989; Hagiwara *et al.*, 1994):

$$g_V^{\nu_\ell} \rightarrow g_V^{\nu_\ell} + \frac{2}{3} m_W^2 \langle r_{\nu_\ell}^2 \rangle \sin^2 \theta_W. \quad (7.23)$$

Using this method, experiments which measure neutrino-electron elastic scattering can probe the neutrino charge radius. Some experimental results are listed in Tab. V. In addition, Hirsch *et al.* (2003) obtained the following 90% CL bounds on $\langle r_{\nu_\mu}^2 \rangle$ from a reanalysis of CHARM-II (Vilain *et al.*, 1995) and CCFR (McFarland *et al.*, 1998)

data:

$$-0.52 \times 10^{-32} < \langle r_{\nu_\mu}^2 \rangle < 0.68 \times 10^{-32} \text{ cm}^2. \quad (7.24)$$

More recently, Barranco *et al.* (2008) obtained the following 90% CL bounds on $\langle r_{\nu_e}^2 \rangle$ from a combined fit of all available $\nu_e e^-$ and $\bar{\nu}_e e^-$ data:

$$-0.26 \times 10^{-32} < \langle r_{\nu_e}^2 \rangle < 6.64 \times 10^{-32} \text{ cm}^2 \quad (7.25)$$

The single photon production process $e^+ e^- \rightarrow \nu + \bar{\nu} + \gamma$ has been used to get bounds on the effective ν_τ charge radius, assuming a negligible contribution of the ν_e and ν_μ charge radii (Altherr and Salati, 1994; Tanimoto *et al.*, 2000; Hirsch *et al.*, 2003). For Dirac neutrinos, Hirsch *et al.* (2003) obtained

$$-5.6 \times 10^{-32} < \langle r_{\nu_\tau}^2 \rangle < 6.2 \times 10^{-32} \text{ cm}^2. \quad (7.26)$$

Comparing the theoretical Standard Model values in Eqs. (7.20)–(7.22) with the experimental limits in Tab. V and those in Eqs. (7.24)–(7.26), one can see that they differ at most by one order of magnitude. Therefore, one may expect that the experimental accuracy will soon reach the value needed to probe the theoretical predictions for the neutrino effective charge radius.

The neutrino charge radius has also some impact on astrophysical phenomena and on cosmology. The limits on the cooling of the Sun and white dwarfs due to the plasmon decay process discussed in Subsection V.D induced by a neutrino charge radius led Dolgov and Zel'dovich (1981) to estimate the respective limits $|\langle r_\nu^2 \rangle| \lesssim 10^{-28} \text{ cm}^2$ and $|\langle r_\nu^2 \rangle| \lesssim 10^{-30} \text{ cm}^2$ for all neutrino flavors. From the cooling of red giants Altherr and Salati (1994) inferred the limit $|\langle r_\nu^2 \rangle| \lesssim 4 \times 10^{-31} \text{ cm}^2$.

If neutrinos are Dirac particles, $e^+ e^-$ annihilations can produce right-handed neutrino-antineutrino pairs through the coupling induced by a neutrino charge radius. This process would affect primordial Big-Bang Nucleosynthesis and the energy release of supernova. From the measured ^4He yield in primordial Big-Bang Nucleosynthesis Grifols and Masso (1987) obtained

$$|\langle r_\nu^2 \rangle| \lesssim 7 \times 10^{-33} \text{ cm}^2, \quad (7.27)$$

and from SN 1987A data Grifols and Masso (1989) obtained

$$\langle r_\nu^2 \rangle \lesssim 2 \times 10^{-33} \text{ cm}^2, \quad (7.28)$$

for all neutrino flavors.

C. Neutrino anapole moment

The anapole form factor is the most mysterious among the neutrino form factors. The notion of an anapole moment for a Dirac particle was introduced by Zel'dovich (1958) after the discovery of parity violation. The anapole form factor was not known before because it violates P. Indeed, taking into account that

$$A_\mu(x) \xrightarrow{P} A^\mu(x_P), \quad (7.29)$$

P is conserved if

$$\Lambda_\mu(q) \xrightarrow{P} \Lambda^\mu(q). \quad (7.30)$$

Using the formulae in Appendix A, one can find that

$$\Lambda_\mu(q) \xrightarrow{P} \gamma^0 \Lambda_\mu(q_P) \gamma^0. \quad (7.31)$$

Using the form-factor expansion in Eq. (3.18), we obtain

$$\begin{aligned} \Lambda_\mu(q) \xrightarrow{P} & \mathbb{f}_Q(q^2) \gamma^\mu - \mathbb{f}_M(q^2) i \sigma^{\mu\nu} q_\nu \\ & - \mathbb{f}_E(q^2) \sigma^{\mu\nu} q_\nu \gamma_5 - \mathbb{f}_A(q^2) (q^2 \gamma^\mu - q^\mu \not{q}) \gamma_5. \end{aligned} \quad (7.32)$$

Hence, parity is violated by the electric and anapole moments. Since the anapole moment conserves CP (and T, as a consequence of CPT symmetry), as shown in Section III.A, it follows that the anapole moment violates also C.

In order to understand the physical characteristics of the anapole moment, we consider its effect in the interactions with external electromagnetic fields. From the last term in Eq. (6.7) one can see that the anapole moment describes an interaction with the current which generates the external electromagnetic fields.

Using the method described in Appendix E, we obtain the helicity-conserving potential

$$V_{h \rightarrow h} = -\mathbf{a} h \frac{m}{E} s^\mu j_\mu, \quad (7.33)$$

which is strongly suppressed for ultrarelativistic neutrinos. In the non-relativistic limit, we obtain

$$V_{h \rightarrow h}^{\text{nr}} \simeq \vec{\mathbf{a}} \cdot \vec{j}, \quad \text{with} \quad \vec{\mathbf{a}} = h \frac{\vec{p}}{|\vec{p}|} \mathbf{a}. \quad (7.34)$$

This is the anapole moment potential that was introduced by Zel'dovich (1958). It is proportional to the longitudinal component of the current.

Considering now the helicity-flipping potential, as shown at the end of Appendix E, we obtain

$$V_{-h \rightarrow h} = \mathbf{a} \frac{m}{E} j_\perp, \quad (7.35)$$

where j_\perp is the component of \vec{j} orthogonal to \vec{p} . For ultrarelativistic neutrinos, the helicity-flipping potential is strongly suppressed, but in the non-relativistic limit we have

$$V_{-h \rightarrow h}^{\text{nr}} \simeq \mathbf{a} j_\perp = |\vec{\mathbf{a}} \times \vec{j}|. \quad (7.36)$$

This potential corresponds to a classical torque (Zel'dovich, 1958) which rotates the spin of the particle, causing periodic changes of the helicity.

The anapole moment is a mysterious quantity which is difficult to understand, because it does not generate interactions with a free electromagnetic field, but only contact interactions with the charge and current density which generates an electromagnetic field. A classical model which can help to visualize the behavior of the anapole moment has been given by Zel'dovich (1958) (see also Bukina *et al.* (1998a)). In this model the anapole is represented by a current-carrying rigid toroidal solenoid. The current generates a magnetic field only inside the toroidal solenoid. Since the solenoid is rigid, there is no external magnetic field which can act on the toroidal solenoid as a whole. The only action on the toroidal solenoid can be generated by a current which passes through the solenoid and interacts with the magnetic field inside. For example, the toroidal solenoid can be immersed in an electrolytic solution which fills also the space inside the solenoid. If a current flows through the electrolytic solution, it interacts with the magnetic field inside the solenoid and generates a torque proportional to the sine of the angle between the direction of the current and the axis of the toroid. In this model the axis of the toroid corresponds to the direction of $\vec{\mathbf{a}}$ in Eqs. (7.34) and the torque corresponds to the helicity-flipping potential in Eq. (7.36).

The neutrino anapole moment contributes to the scattering of neutrinos with charged particles. In order to discuss its effects, it is convenient to consider strictly neutral neutrinos with $\mathbb{f}_Q(0) = 0$ and define a reduced charge form factor $\tilde{\mathbb{f}}_Q(q^2)$ such that

$$\mathbb{f}_Q(q^2) = q^2 \tilde{\mathbb{f}}_Q(q^2). \quad (7.37)$$

Then, from Eq. (7.18), apart from a factor 1/6, the reduced charge form factor at $q^2 = 0$ is just the squared neutrino charge radius:

$$\tilde{\mathbb{f}}_Q(0) = \langle r^2 \rangle / 6. \quad (7.38)$$

Let us now consider the charge and anapole parts of the neutrino electromagnetic vertex function in Eq. (3.37), which can be written as

$$\Lambda_\mu^{Q,A}(q) = (\gamma_\mu q^2 - q_\mu \not{q}) \left[\tilde{\mathbb{f}}_Q(q^2) + \mathbb{f}_A(q^2) \gamma_5 \right]. \quad (7.39)$$

Since for ultrarelativistic neutrinos the effect of γ_5 is only a sign which depends on the helicity of the neutrino (see Eq. (C6)), the phenomenology of neutrino anapole moments is similar to that of neutrino charge radii. Hence, the limits on the neutrino charge radii discussed in Subsection VII.B apply also to the neutrino anapole moments multiplied by 6.

As we have seen in Subsection VII.A, in the Standard Model the neutrino electric charges are exactly zero. Hence, Eqs. (7.39) applies to Standard Model and can be further simplified taking into account that in the Standard Model neutrinos are described by two-component massless left-handed Weyl spinors. As discussed in Subsection III.C, the γ^5 in Eq. (7.39) becomes a minus sign, leading to

$$\Lambda_{\text{SM}\mu}^{Q,A}(q) = (\gamma_\mu q^2 - q_\mu \not{q}) \mathbb{f}^{\text{SM}}(q^2), \quad (7.40)$$

with

$$\mathbb{f}^{\text{SM}}(q^2) = \tilde{\mathbb{f}}_Q(q^2) - \mathbb{f}_A(q^2) \xrightarrow{q^2 \rightarrow 0} \frac{\langle r^2 \rangle}{6} - \mathbf{a}. \quad (7.41)$$

These equations correspond to Eqs. (3.83) and (3.84) for $\mathbb{f}_Q(0) = 0$. Hence, in the Standard Model the neutrino charge radius and the anapole moment are not defined separately and one can interpret arbitrarily $\mathbb{f}^{\text{SM}}(0)$ as a charge radius or as an anapole moment. This is the correct interpretation of the statement often found in the literature that in the Standard Model $\mathbf{a} = -\langle r^2 \rangle/6$. Note also that it follows that the Standard Model values for the neutrino charge radii in Eqs. (7.19)–(7.22) can be interpreted as values of the corresponding neutrino anapole moments.

Some deep insight into an interpretation of the decompositions of the vertex function (3.18) and the neutrino form factors can be obtained in the framework of a multipole expansions of the corresponding classical electromagnetic currents (Dubovik and Cheshkov, 1974; Dubovik and Tosunian, 1983; Dubovik and Kuznetsov, 1998). Since in this limit the anapole form factor does not correspond to a certain multipole distribution (that is why the term “anapole” was introduced by Zel’dovich (1958)), the anapole moment has a quite intricate classical analog. Therefore, Dubovik and Kuznetsov (1998); Bukina *et al.* (1998a,b) proposed to consider the toroidal dipole moment as a characteristic of the neutrino which is more convenient and transparent than the anapole moment for the description of T-invariant interactions with non-conservation of the P and C symmetries. In this case, the electromagnetic vertex of a neutrino can be rewritten in the alternative multipole (toroidal) parameterization

$$\Lambda_\mu(q) = \mathbb{f}_Q(q^2)\gamma_\mu - \mathbb{f}_M(q^2)i\sigma_{\mu\nu}q^\nu + \mathbb{f}_E(q^2)\sigma_{\mu\nu}q^\nu\gamma_5 + i\mathbb{f}_T(q^2)\epsilon_{\mu\nu\lambda\rho}P^\nu q^\lambda\gamma^\rho, \quad (7.42)$$

where \mathbb{f}_T is the toroidal dipole form factor and $P = p_i +$

p_f . From the following identity

$$\bar{u}_f(p_f) \left\{ (m_i - m_f)\sigma_{\mu\nu}q^\nu + (q^2\gamma_\mu - \not{q}q_\mu) - i\varepsilon_{\mu\nu\lambda\rho}P^\nu q^\lambda\gamma^\rho\gamma_5 \right\} \gamma_5 u_i(p_i) = 0, \quad (7.43)$$

it can be seen that the toroidal and anapole moments coincide in the static limit when the masses of the initial and final neutrino states are equal to each other, $m_i = m_f$ (Bukina *et al.*, 1998b), i.e. the toroidal and anapole parameterizations coincide in this case.

In some sense the toroidal parameterization has a more transparent and clear physical interpretation, because it provides a one-to-one correspondence between the multipole moments and the corresponding form factors. From the properties of each term in the expression (7.42) for the vertex function under C, P and T transformations, it follows that in the Majorana case only the toroidal form factor survives (Zel’dovich, 1958; Kobzarev and Okun, 1972) and the toroidal moment of the Dirac neutrino is half of that in the Majorana case.

In one-loop calculations (Dubovik and Kuznetsov, 1998) of the toroidal (and anapole) moment of a massive and a massless Majorana neutrino (the diagrams in Figs. 6 and 7 contribute) it was shown that its value does not depend significantly on the neutrino mass (through the ratios $m_{\nu_i}^2/m_W^2$) and is of the order of

$$\mathbb{f}_T(q^2 = 0) \sim e \times (10^{-33} - 10^{-34}) \text{ cm}^2, \quad (7.44)$$

depending on the values of the quark masses that propagate in the loop diagrams in Fig. 8.

Note that the toroidal form factors can contribute to the neutrino vertex function in both the diagonal and off-diagonal cases.

The toroidal (anapole) interactions of a Majorana as well as a Dirac neutrino are expected to contribute to the total cross section of neutrino elastic scattering off electrons, quarks and nuclei. Due to the fact that the toroidal (anapole) interactions contribute to the helicity preserving part of the scattering of neutrinos on electrons, quarks and nuclei, its contribution to cross sections are similar to those of the neutrino charge radius. In principle, these contributions can be probed and information about toroidal moments can be extracted in low-energy scattering experiments in the future.

Different effects of the neutrino toroidal moment are discussed in Ginzburg and Tsytovich (1985); Dubovik and Kuznetsov (1998); Bukina *et al.* (1998a,b). In particular, it has been shown that the neutrino toroidal electromagnetic interactions can produce Cherenkov radiation of neutrinos propagating in a medium.

VIII. SUMMARY AND PERSPECTIVES

In this review we discussed the theory and phenomenology of neutrino electromagnetic properties and

interactions. We have seen that most of the theoretical and experimental research has been devoted to the study of magnetic and electric dipole moments, but there has been also some interest in the investigation of neutrino millicharges and of the charge radii and anapole moments of neutrinos.

Unfortunately, so far there is not any experimental indication in favor of neutrino electromagnetic interactions and all neutrino electromagnetic properties are known to be small, with rather stringent upper bounds obtained in laboratory experiments or from astrophysical observations.

The most accessible neutrino electromagnetic property may be the charge radius, discussed in Section VII.B, for which the Standard Model gives a value which is only about one order of magnitude smaller than the experimental upper bounds. A measurement of a neutrino charge radius at the level predicted by the Standard Model would be another spectacular confirmation of the Standard Model, after the recent discovery of the Higgs boson (see Ellis (2013)). However, such a measurement would not give information on new physics beyond the Standard Model unless the measured value is shown to be incompatible with the Standard Model value in a high-precision experiment.

The strongest current efforts to probe the physics beyond the Standard Model by measuring neutrino electromagnetic properties is the search for a neutrino magnetic moment effect in reactor $\bar{\nu}_e$ - e^- scattering experiments. The current upper bounds reviewed in Section IV.E are more than eight orders of magnitude larger than the prediction discussed in Section IV.A of the Dirac neutrino magnetic moments in the minimal extension of the Standard Model with right-handed neutrinos. Hence, a discovery of a neutrino magnetic moment effect in reactor $\bar{\nu}_e$ - e^- scattering experiments would be a very exciting discovery of non-minimal new physics beyond the Standard Model.

In particular, the GEMMA-II collaboration expects to reach around the year 2017 a sensitivity to $\mu_{\nu_e} \approx 1 \times 10^{-11} \mu_B$ in a new series of measurements at the Kalinin Nuclear Power Plant with a doubled neutrino flux obtained by reducing the distance between the reactor and the detector from 13.9 m to 10 m and by reducing the energy threshold from 2.8 keV to 1.5 keV (Beda *et al.*, 2012, 2013). The corresponding sensitivity to the neutrino electric millicharge discussed in Section VII.A will reach the level of $|q_{\nu_e}| \approx 3.7 \times 10^{-13} e$ (Studenikin, 2013).

There is also a GEMMA-III project to further lower the energy threshold to about 350 eV, which may allow the experimental collaboration to reach a sensitivity to $\mu_{\nu_e} \approx 9 \times 10^{-12} \mu_B$ within the year 2020¹³. The

corresponding sensitivity to neutrino millicharge will be $|q_{\nu_e}| \approx 1.8 \times 10^{-13} e$ (Studenikin, 2013).

A further interesting new possibility for exploring very small values of μ_{ν_e} in $\bar{\nu}_e$ - e^- scattering experiments has been proposed by (Bernabeu *et al.*, 2005b) on the basis of the observation (Segura *et al.*, 1994) that “dynamical zeros” induced by a destructive interference between the left-handed and right-handed chiral couplings of the electron in the charged and neutral current amplitudes appear in the Standard Model contribution to the scattering cross section. Selecting recoil electrons contained in a forward narrow cone corresponding to a dynamical zero (see Eq. (4.23)) it may be possible to enhance the sensitivity of an experiment to μ_{ν_e} .

Let us finally emphasize that neutrino electromagnetic interactions could have important effects in astrophysical environments and in the evolution of the Universe and the current rapid advances of astrophysical and cosmological observations may lead soon to the very exciting discovery of non-standard neutrino electromagnetic properties.

Appendix A: Useful constants and formulae

In this Appendix we list some useful physical constants and formulae used in the paper.

We use natural units in which $c = \hbar = 1$, where c is the velocity of light and \hbar is the reduced Planck constant.

The values of the following physical constants are taken

¹³ Dmitry Medvedev, private communication.

from Beringer *et al.* (2012).

Avogadro number:

$$N_A = 6.022\,141\,29\,(27) \times 10^{23} \text{ mol}^{-1}. \quad (\text{A1})$$

Bohr magneton ($\mu_B \equiv e/2m_e$):

$$\begin{aligned} \mu_B &= 5.788\,381\,8066\,(38) \times 10^{-15} \text{ MeV G}^{-1} \\ &\simeq 0.296 \text{ MeV}^{-1}. \end{aligned} \quad (\text{A2})$$

Electron mass:

$$m_e = 0.510\,998\,928\,(11) \text{ MeV}. \quad (\text{A3})$$

Conversion constant:

$$\hbar c = 1.973\,269\,718\,(44) \times 10^{-5} \text{ eV cm}. \quad (\text{A4})$$

Light velocity:

$$c = 299\,792\,458 \text{ m s}^{-1}. \quad (\text{A5})$$

Fermi constant:

$$G_F = 1.166\,378\,7\,(6) \times 10^{-5} \text{ GeV}^{-2}. \quad (\text{A6})$$

Fine-structure constant ($\alpha \equiv e^2/4\pi$) at $Q^2 = 0$:

$$\alpha^{-1} = 137.035\,999\,074\,(44). \quad (\text{A7})$$

Neutron mass:

$$m_n = 939.565\,379\,(21) \text{ MeV}. \quad (\text{A8})$$

Proton mass:

$$m_p = 938.272\,046\,(21) \text{ MeV}. \quad (\text{A9})$$

Muon mass:

$$m_\mu = 105.658\,3715\,(35) \text{ MeV}. \quad (\text{A10})$$

Planck constant, reduced:

$$\hbar = 6.582\,119\,28\,(15) \times 10^{-22} \text{ MeV s}. \quad (\text{A11})$$

Tau mass:

$$m_\tau = 1776.82\,(16) \text{ MeV}. \quad (\text{A12})$$

Weak mixing angle (on-shell: $\sin^2 \theta_W \equiv 1 - m_W^2/m_Z^2$):

$$\sin^2 \theta_W = 0.222\,95\,(28). \quad (\text{A13})$$

For Dirac γ matrices and related quantities we use the notation and conventions in Giunti and Kim (2007). Curly and square brackets denote, respectively, anti-commutator and commutator. For a four-vector p^μ we use the standard notation $\not{p} \equiv p^\mu \gamma_\mu$, with the metric tensor $g^{\mu\nu} = g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and $p^\mu = (p^0, p^1, p^2, p^3) = (p^0, \vec{p})$.

Dirac γ matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad (\text{A14})$$

$$\gamma^0 \gamma^\mu \gamma^0 = \gamma^{\mu\dagger} = \gamma_\mu. \quad (\text{A15})$$

Definition and properties of γ^5 :

$$\gamma^5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma_5, \quad (\text{A16})$$

$$\{\gamma^5, \gamma^\mu\} = 0, \quad (\text{A17})$$

$$(\gamma^5)^2 = \mathbb{1}, \quad (\text{A18})$$

$$(\gamma^5)^\dagger = \gamma^5, \quad (\text{A19})$$

$$\gamma^\mu \gamma^5 = \frac{i}{6} \epsilon^{\mu\nu\rho\sigma} \gamma_\nu \gamma_\rho \gamma_\sigma. \quad (\text{A20})$$

Definition and properties of $\sigma^{\mu\nu}$:

$$\sigma^{\mu\nu} \equiv \frac{i}{2} [\gamma^\mu, \gamma^\nu] = i \gamma^\mu \gamma^\nu - i \gamma^{\mu\nu}, \quad (\text{A21})$$

$$[\gamma^5, \sigma^{\mu\nu}] = 0, \quad (\text{A22})$$

$$\gamma^0 \sigma^{\mu\nu} \gamma^0 = (\sigma^{\mu\nu})^\dagger = \sigma_{\mu\nu}, \quad (\text{A23})$$

$$\epsilon^{\mu\nu\alpha\beta} \sigma_{\alpha\beta} = -2i \sigma^{\mu\nu} \gamma^5, \quad (\text{A24})$$

$$\epsilon^{\mu\nu\alpha\beta} \gamma_\nu = i (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) \gamma_\nu \gamma^5 - \gamma^\mu \sigma^{\alpha\beta} \gamma^5. \quad (\text{A25})$$

Definition and properties of $\vec{\Sigma}$:

$$\Sigma^k \equiv \frac{1}{2} \sum_{j,l} \epsilon^{kjl} \sigma^{jl} = \gamma^0 \gamma^k \gamma^5, \quad (\text{A26})$$

$$[\Sigma^k, \Sigma^j] = 2i \sum_l \epsilon^{kjl} \Sigma^l, \quad (\text{A27})$$

$$\{\Sigma^k, \Sigma^j\} = 2\delta^{kj}, \quad (\text{A28})$$

$$(\Sigma^k)^\dagger = \Sigma^k, \quad (\text{A29})$$

$$[\Sigma^k, \gamma^0] = [\Sigma^k, \gamma^5] = 0, \quad (\text{A30})$$

$$[\Sigma^k, \gamma^j] = 2i \sum_\ell \epsilon^{kj\ell} \gamma^\ell. \quad (\text{A31})$$

Charge-conjugation matrix:

$$\mathcal{C} \gamma_\mu^T \mathcal{C}^{-1} = -\gamma_\mu, \quad (\text{A32})$$

$$\mathcal{C}^\dagger = \mathcal{C}^{-1}, \quad (\text{A33})$$

$$\mathcal{C}^T = -\mathcal{C}. \quad (\text{A34})$$

$$\mathcal{C} (\gamma^5)^T \mathcal{C}^{-1} = \gamma^5, \quad (\text{A35})$$

$$\mathcal{C} (\sigma^{\mu\nu})^T \mathcal{C}^{-1} = -\sigma^{\mu\nu}. \quad (\text{A36})$$

Traces of products of γ matrices:

$$\text{Tr}[\gamma^\alpha \gamma^\beta] = 4g^{\alpha\beta}, \quad (\text{A37})$$

$$\text{Tr}[\gamma^\alpha \gamma^\beta \gamma^\rho \gamma^\sigma] = 4(g^{\alpha\beta} g^{\rho\sigma} - g^{\alpha\rho} g^{\beta\sigma} + g^{\alpha\sigma} g^{\beta\rho}), \quad (\text{A38})$$

$$\text{Tr}[\gamma^\alpha \gamma^\beta \gamma^\rho \gamma^\sigma \gamma^5] = -4i \epsilon^{\alpha\beta\rho\sigma}. \quad (\text{A39})$$

The γ matrices are traceless. The trace of a product of an odd number of γ matrices is zero. $\text{Tr}[\gamma^5] = \text{Tr}[\gamma^\alpha \gamma^\beta \gamma^5] = 0$.

Four-momentum and helicity eigenstate spinors:

$$(\not{p} - m) u^{(h)}(p) = 0, \quad (\text{A40})$$

$$(\not{p} + m) v^{(h)}(p) = 0. \quad (\text{A41})$$

Normalization:

$$\overline{u^{(h)}}(p) u^{(h')}(p) = 2m \delta_{hh'}, \quad (\text{A42})$$

$$\overline{v^{(h)}}(p) v^{(h')}(p) = -2m \delta_{hh'}, \quad (\text{A43})$$

Charge-conjugation relation:

$$u^{(h)}(p) = \mathcal{C} \overline{v^{(h)}}^T(p) \quad (\text{A44})$$

$$v^{(h)}(p) = \mathcal{C} \overline{u^{(h)}}^T(p). \quad (\text{A45})$$

Energy-projection matrices:

$$\Lambda_+(p) = \frac{m + \not{p}}{2m} = \sum_{h=\pm 1} \frac{u^{(h)}(p) \overline{u^{(h)}}(p)}{2m}, \quad (\text{A46})$$

$$\Lambda_-(p) = \frac{m - \not{p}}{2m} = - \sum_{h=\pm 1} \frac{v^{(h)}(p) \overline{v^{(h)}}(p)}{2m}. \quad (\text{A47})$$

Helicity-projection matrices ($[P_h, \Lambda_\pm(p)] = 0$):

$$P_h = \frac{1 + h \gamma^5 \not{s}}{2}, \quad (\text{A48})$$

for $h = +1$ (right-handed) and $h = -1$ (left-handed), with the polarization four-vector

$$s^\mu = \left(\frac{|\vec{p}|}{m}, \frac{E}{m} \frac{\vec{p}}{|\vec{p}|} \right), \quad s^2 = -1, \quad s \cdot p = 0. \quad (\text{A49})$$

The helicity eigenstate spinors satisfy

$$P_{h'} u^{(h)}(p) = \frac{1}{2} \left(1 + h' \frac{\vec{p} \cdot \vec{\Sigma}}{|\vec{p}|} \right) u^{(h)}(p) = \delta_{h'h} u^{(h)}(p), \quad (\text{A50})$$

$$P_{h'} v^{(h)}(p) = \frac{1}{2} \left(1 - h' \frac{\vec{p} \cdot \vec{\Sigma}}{|\vec{p}|} \right) v^{(h)}(p) = \delta_{h'h} v^{(h)}(p). \quad (\text{A51})$$

Moreover, since $\gamma^0 P_h^\dagger \gamma^0 = P_h$, we also have

$$\overline{u^{(h)}}(p) P_{h'} = \delta_{h'h} \overline{u^{(h)}}(p), \quad \overline{v^{(h)}}(p) P_{h'} = \delta_{h'h} \overline{v^{(h)}}(p). \quad (\text{A52})$$

Fourier expansion of a free Dirac field $\psi(x)$ (with $p^0 = E = \sqrt{\vec{p}^2 + m^2}$):

$$\psi(x) = \int \frac{d^3 p}{(2\pi)^3 2E} \sum_{h=\pm 1} \left[a^{(h)}(p) u^{(h)}(p) e^{-ip \cdot x} + b^{(h)\dagger}(p) v^{(h)}(p) e^{ip \cdot x} \right]. \quad (\text{A53})$$

The operators $a^{(h)}(p)$, $a^{(h)\dagger}(p)$, $b^{(h)}(p)$, $b^{(h)\dagger}(p)$ anticommute, except for

$$\{a^{(h)}(p), a^{(h')\dagger}(p')\} = \{b^{(h)}(p), b^{(h')\dagger}(p')\} = (2\pi)^3 2E \delta^3(\vec{p} - \vec{p}') \delta_{hh'}. \quad (\text{A54})$$

States describing a fermion f and an antifermion \bar{f} with four-momentum p and helicity h ($|0\rangle$ is the vacuum, such that $a^{(h)}(p)|0\rangle = 0$, $b^{(h)}(p)|0\rangle = 0$ and $\langle 0|0\rangle = 1$; V is the total volume):

$$|f(p, h)\rangle = a^{(h)\dagger}(p) |0\rangle, \quad |\bar{f}(p, h)\rangle = b^{(h)\dagger}(p) |0\rangle, \quad (\text{A55})$$

$$\langle f(p, h) | f(p, h') \rangle = \langle \bar{f}(p, h) | \bar{f}(p, h') \rangle = 2E V \delta_{hh'}. \quad (\text{A56})$$

The Majorana condition (2.25) leads to the following Fourier expansion of a free Majorana field:

$$\psi(x) = \int \frac{d^3 p}{(2\pi)^3 2E} \sum_{h=\pm 1} \left[a^{(h)}(p) u^{(h)}(p) e^{-ip \cdot x} + a^{(h)\dagger}(p) v^{(h)}(p) e^{ip \cdot x} \right]. \quad (\text{A57})$$

Gordon identities:

$$\overline{u_f}(p_f) i\sigma^{\alpha\beta} (p_f - p_i)_\beta u_i(p_i) = \overline{u_f}(p_f) [(m_f + m_i) \gamma^\alpha - (p_f + p_i)^\alpha] u_i(p_i), \quad (\text{A58})$$

$$\overline{u_f}(p_f) i\sigma^{\alpha\beta} (p_f + p_i)_\beta u_i(p_i) = \overline{u_f}(p_f) [(m_f - m_i) \gamma^\alpha - (p_f - p_i)^\alpha] u_i(p_i), \quad (\text{A59})$$

$$\overline{u_f}(p_f) i\sigma^{\alpha\beta} (p_f - p_i)_\beta \gamma^5 u_i(p_i) = \overline{u_f}(p_f) [(m_f - m_i) \gamma^\alpha - (p_f + p_i)^\alpha] \gamma^5 u_i(p_i), \quad (\text{A60})$$

$$\overline{u_f}(p_f) i\sigma^{\alpha\beta} (p_f + p_i)_\beta \gamma^5 u_i(p_i) = \overline{u_f}(p_f) [(m_f + m_i) \gamma^\alpha - (p_f - p_i)^\alpha] \gamma^5 u_i(p_i). \quad (\text{A61})$$

C, P, CP, T and CPT active transformations of a fermionic field $\psi(x)$ ($x_P^\mu = x_\mu$ and $x_T^\mu = -x_\mu$):

$$U_C \psi(x) U_C^\dagger = \xi^C \mathcal{C} \overline{\psi}^T(x), \quad (\text{A62})$$

$$U_P \psi(x) U_P^\dagger = \xi^P \gamma^0 \psi(x_P), \quad (\text{A63})$$

$$U_{CP} \psi(x) U_{CP}^\dagger = \xi^{CP} \gamma^0 \mathcal{C} \overline{\psi}^T(x_P), \quad (\text{A64})$$

$$U_T \psi(x) U_T^\dagger = \xi^T \mathcal{C}^\dagger \gamma^5 \psi(x_T), \quad (\text{A65})$$

$$U_{CPT} \psi(x) U_{CPT}^\dagger = \xi^{CPT} [\psi^\dagger(-x) \gamma^5]^T. \quad (\text{A66})$$

Here ξ^C , ξ^P , ξ^{CP} , ξ^T , ξ^{CPT} are phases such that $\xi^{CP} = \xi^C \xi^P$, $\xi^T = \pm \xi^{CP}$ or $\pm i \xi^{CP}$, and $\xi^{CPT} = \xi^T \xi^{CP*} = \pm 1$ or $\pm i$. The operators U_C , U_P and U_{CP} are unitary, whereas the operators U_T and U_{CPT} are antiunitary. An antiunitary operator U satisfies the standard relation $U^\dagger = U^{-1}$ of unitary operators, but is antilinear, i.e. for a real number z

$$U z U^\dagger = z^*, \quad (\text{A67})$$

and if $|p'\rangle = U|p\rangle$ we have

$$\langle p'_1 | p'_2 \rangle = \langle p_1 | U^\dagger U | p_2 \rangle^* = \langle p_2 | p_1 \rangle. \quad (\text{A68})$$

Maxwell equations in the International System of Units (SI):

$$\partial_\mu F^{\mu\nu}(x) = j^\nu(x), \quad \partial_\mu \tilde{F}^{\mu\nu}(x) = 0, \quad (\text{A69})$$

where $j^\mu(x) = (\rho(x), \vec{j}(x))$ is the four-vector of the charge and current density and

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x), \quad (\text{A70})$$

$$\tilde{F}^{\mu\nu}(x) \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}(x), \quad (\text{A71})$$

where $A^\mu(x)$ is the electromagnetic field. The electromagnetic tensor $F^{\mu\nu}(x)$ contains the physical electric field $\vec{E}(x)$ and magnetic field $\vec{B}(x)$:

$$E^k(x) = F^{k0}(x), \quad B^k(x) = -\frac{1}{2}\sum_{j,\ell}\epsilon^{kj\ell}F^{j\ell}(x). \quad (\text{A72})$$

The Maxwell equations for the electromagnetic field are ($\square \equiv \partial_\mu \partial^\mu$):

$$\square A^\mu(x) - \partial^\mu \partial_\nu A^\nu(x) = j^\mu(x). \quad (\text{A73})$$

Appendix B: Decomposition of Λ_μ

In this Appendix (see also Nowakowski *et al.* (2005)) we derive the general expression of $\Lambda_\mu(p_i, p_f)$ in the matrix element

$$\langle \nu_f(p_f) | j_\mu^{(\nu)}(0) | \nu_i(p_i) \rangle = \overline{u}_f(p_f) \Lambda_\mu(p_i, p_f) u_i(p_i). \quad (\text{B1})$$

The initial and final massive neutrinos can be different, but since they are considered as free particles they are on shell, with four-momenta p_i and p_f such that

$$p_i^2 = m_i^2, \quad p_f^2 = m_f^2, \quad (\text{B2})$$

We use the notation

$$q \equiv p_i - p_f, \quad t \equiv p_i + p_f, \quad (\text{B3})$$

for which we have

$$q^2 + t^2 = 2(m_f^2 + m_i^2), \quad q \cdot t = m_i^2 - m_f^2. \quad (\text{B4})$$

In general, $\Lambda^\mu(q, t)$ can be expanded as a linear combination of the 16 matrices

$$\mathbb{1}, \quad \gamma^\mu, \quad \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu], \quad \gamma^\mu \gamma^5, \quad \gamma^5, \quad (\text{B5})$$

which form a basis in the vectorial space of 4×4 matrices (see Giunti and Kim (2007)). Since $\Lambda^\mu(q, t)$ carries a Lorentz index, the coefficients of this expansion can depend on the available tensors: the four-vectors q^μ , t^μ , the Lorentz-invariant metric tensor $g^{\mu\nu}$ and the Lorentz-invariant antisymmetric tensor $\epsilon^{\mu\nu\alpha\beta}$. Let us consider separately each term of the expansion:

1. The $\mathbb{1}$ term is a linear combination of the set

$$S(\mathbb{1}) = \{q^\mu \mathbb{1}, t^\mu \mathbb{1}\}. \quad (\text{B6})$$

2. The γ^μ term is a linear combination of the set

$$S(\gamma^\mu) = \{\gamma^\mu, q^\mu \not{q}, q^\mu \not{t}, t^\mu \not{q}, t^\mu \not{t}, \epsilon^{\mu\nu\alpha\beta} \gamma_\nu q_\alpha t_\beta\}. \quad (\text{B7})$$

3. The $\sigma^{\mu\nu}$ term is a linear combination of the set

$$S(\sigma^{\mu\nu}) = \{\sigma^{\mu\nu} q_\nu, \sigma^{\mu\nu} t_\nu, q^\mu \sigma^{\alpha\beta} q_\alpha t_\beta, t^\mu \sigma^{\alpha\beta} q_\alpha t_\beta, \epsilon^{\mu\nu\alpha\beta} q_\nu \sigma_{\alpha\beta}, \epsilon^{\mu\nu\alpha\beta} t_\nu \sigma_{\alpha\beta}, \epsilon^{\mu\nu\alpha\beta} \sigma_{\nu\rho} q_\alpha t_\beta q^\rho, \epsilon^{\mu\nu\alpha\beta} \sigma_{\nu\rho} q_\alpha t_\beta t^\rho\}. \quad (\text{B8})$$

4. The $\gamma^\mu \gamma^5$ term is a linear combination of the set

$$S(\gamma^\mu \gamma^5) = \{\gamma^\mu \not{q} \gamma^5, q^\mu \not{q} \gamma^5, q^\mu \not{t} \gamma^5, t^\mu \not{q} \gamma^5, t^\mu \not{t} \gamma^5, \epsilon^{\mu\nu\alpha\beta} \gamma_\nu q_\alpha t_\beta \gamma^5\}. \quad (\text{B9})$$

5. The γ^5 term is a linear combination of the set

$$S(\gamma^5) = \{q^\mu \gamma^5, t^\mu \gamma^5\}. \quad (\text{B10})$$

Several elements of the sets (B6)–(B10) can be expressed as linear combinations of others, leading to only six independent elements. It is convenient to choose the set of six independent elements as

$$q^\mu \mathbb{1}, \quad q^\mu \gamma^5, \quad \gamma^\mu, \quad \gamma^\mu \gamma^5, \quad \sigma^{\mu\nu} q_\nu, \quad \epsilon^{\mu\nu\alpha\beta} q_\nu \sigma_{\alpha\beta}. \quad (\text{B11})$$

We express all the elements in the sets (B6)–(B10) in terms of the six elements in the set (B11) using the equations in Appendix A as follows (above each arrow we indicate the main equations used in the decomposition):

1. From $S(\mathbb{1})$:

$$t^\mu \mathbb{1} \xrightarrow{(\text{A58})} \{\gamma^\mu, \sigma^{\mu\nu} q_\nu\}. \quad (\text{B12})$$

2. From $S(\gamma^\mu)$:

$$q^\mu \not{q} \xrightarrow{(\text{A40})} \{q^\mu \mathbb{1}\}, \quad (\text{B13})$$

$$q^\mu \not{t} \xrightarrow{(\text{A40})} \{q^\mu \mathbb{1}\}, \quad (\text{B14})$$

$$t^\mu \not{q} \xrightarrow{(\text{A58})} \{\gamma^\mu, \sigma^{\mu\nu} q_\nu\}, \quad (\text{B15})$$

$$t^\mu \not{t} \xrightarrow{(\text{A58})} \{\gamma^\mu, \sigma^{\mu\nu} q_\nu\}, \quad (\text{B16})$$

$$\epsilon^{\mu\nu\alpha\beta} \gamma_\nu q_\alpha t_\beta \xrightarrow{(\text{A25})+(\text{A60})} \{q^\mu \gamma^5, \gamma^\mu \gamma^5, \epsilon^{\mu\nu\alpha\beta} q_\nu \sigma_{\alpha\beta}\}. \quad (\text{B17})$$

3. From $S(\sigma^{\mu\nu})$:

$$\sigma^{\mu\nu} t_\nu \xrightarrow{(\text{A59})} \{q^\mu \mathbb{1}, \gamma^\mu\}, \quad (\text{B18})$$

$$q^\mu \sigma^{\alpha\beta} q_\alpha t_\beta \xrightarrow{(\text{A59})} \{q^\mu \mathbb{1}\}, \quad (\text{B19})$$

$$t^\mu \sigma^{\alpha\beta} q_\alpha t_\beta \xrightarrow{(\text{A58})} \{\gamma^\mu, \sigma^{\mu\nu} q_\nu\}, \quad (\text{B20})$$

$$\epsilon^{\mu\nu\alpha\beta} t_\nu \sigma_{\alpha\beta} \xrightarrow{(\text{A61})} \{q^\mu \gamma^5, \gamma^\mu \gamma^5\}, \quad (\text{B21})$$

$$\epsilon^{\mu\nu\alpha\beta} \sigma_{\nu\rho} q_\alpha t_\beta q^\rho \xrightarrow{(\text{A25})+(\text{A60})} \{q^\mu \gamma^5, \gamma^\mu \gamma^5, \epsilon^{\mu\nu\alpha\beta} q_\nu \sigma_{\alpha\beta}\}, \quad (\text{B22})$$

$$\epsilon^{\mu\nu\alpha\beta} \sigma_{\nu\rho} q_\alpha t_\beta t^\rho \xrightarrow{(\text{A25})+(\text{A60})} \{q^\mu \gamma^5, \gamma^\mu \gamma^5, \epsilon^{\mu\nu\alpha\beta} q_\nu \sigma_{\alpha\beta}\}. \quad (\text{B23})$$

4. From $S(\gamma^\mu \gamma^5)$:

$$q^\mu \not{q} \gamma^5 \xrightarrow{(A40)} \{q^\mu \gamma^5\}, \quad (B24)$$

$$q^\mu \not{\not{q}} \gamma^5 \xrightarrow{(A40)} \{q^\mu \gamma^5\}, \quad (B25)$$

$$t^\mu \not{q} \gamma^5 \xrightarrow{(A60)} \{\gamma^\mu \gamma^5, \epsilon^{\mu\nu\alpha\beta} q_\nu \sigma_{\alpha\beta}\}, \quad (B26)$$

$$t^\mu \not{\not{q}} \gamma^5 \xrightarrow{(A60)} \{\gamma^\mu \gamma^5, \epsilon^{\mu\nu\alpha\beta} q_\nu \sigma_{\alpha\beta}\}, \quad (B27)$$

$$\epsilon^{\mu\nu\alpha\beta} \gamma_\nu q_\alpha t_\beta \gamma^5 \xrightarrow{(A25)+(A58)} \{q^\mu \mathbb{1}, \gamma^\mu, \sigma^{\mu\nu} q_\nu\}. \quad (B28)$$

5. From $S(\gamma^5)$:

$$t^\mu \gamma^5 \xrightarrow{(A60)} \{\gamma^\mu \gamma^5, \epsilon^{\mu\nu\alpha\beta} q_\nu \sigma_{\alpha\beta}\}. \quad (B29)$$

Appendix C: Helicity and chirality

In this Appendix we derive the relation between helicity and chirality for ultrarelativistic neutrinos and the corresponding helicity conservation properties of the different terms of the general expansion of $\Lambda_\mu^{fi}(q)$ in Eq. (3.35), taking into account that $\Lambda_\mu^{fi}(q)$ is sandwiched between u -spinors in the case of neutrinos (Eq. (3.28)) or between v -spinors in the case of antineutrinos (Eq. (3.45)).

With the help of Eqs. (A40) and (A41) and using the definition (A49) of the polarization four-vector s^μ , one can find that

$$\not{s} u(p) = \left(-\frac{m}{|\vec{p}|} \gamma^0 + \frac{E}{|\vec{p}|} \right) u(p), \quad (C1)$$

$$\not{s} v(p) = \left(-\frac{m}{|\vec{p}|} \gamma^0 - \frac{E}{|\vec{p}|} \right) v(p). \quad (C2)$$

Therefore, in the ultrarelativistic limit $m \ll E$ we have

$$\not{s} u(p) \simeq u(p) \quad \text{and} \quad \not{s} v(p) \simeq -v(p), \quad (C3)$$

and the helicity-projection matrices in Eq. (A48) have the same effect as the chirality projection matrices:

$$P_h u^{(h)}(p) \simeq \frac{1 + h\gamma^5}{2} u^{(h)}(p), \quad (C4)$$

$$P_h v^{(h)}(p) \simeq \frac{1 - h\gamma^5}{2} v^{(h)}(p). \quad (C5)$$

Then, Eqs. (A50) and (A51) imply that $u^{(h)}(p)$ and $v^{(h)}(p)$ are approximate eigenstates of γ^5 :

$$\gamma^5 u^{(h)}(p) \simeq h u^{(h)}(p), \quad (C6)$$

$$\gamma^5 v^{(h)}(p) \simeq -h v^{(h)}(p). \quad (C7)$$

Hence, in the ultrarelativistic limit we have

$$\begin{aligned} \overline{u_f^{(h_f)}} \gamma^\mu \gamma^5 u_i^{(h_i)} &\simeq \overline{h_i u_f^{(h_f)}} \gamma^\mu u_i^{(h_i)} \\ &\simeq h_f \overline{u_f^{(h_f)}} \gamma^\mu u_i^{(h_i)} \propto \delta_{h_f h_i}, \end{aligned} \quad (C8)$$

$$\begin{aligned} \overline{u_f^{(h_f)}} \sigma^{\mu\nu} \gamma^5 u_i^{(h_i)} &\simeq \overline{h_i u_f^{(h_f)}} \sigma^{\mu\nu} u_i^{(h_i)} \\ &\simeq -h_f \overline{u_f^{(h_f)}} \sigma^{\mu\nu} u_i^{(h_i)} \propto \delta_{-h_f h_i}, \end{aligned} \quad (C9)$$

and similar relations hold true in the case of v -spinors. Of course, in the limit of massless neutrinos all the approximations above become exact equalities.

Therefore, in the ultrarelativistic limit the interactions generated by the charge and anapole form factors conserve helicity, whereas the interactions generated by the electric and magnetic dipole form factors flip helicity. In the same way, one can see that the weak interactions generated by the charged current (2.26) or by the neutral current (2.33) conserve helicity in the ultrarelativistic limit.

Appendix D: Calculation of atomic ionization

Consider the process where a neutrino with energy-momentum $p_\nu = (E_\nu, \vec{p}_\nu)$ scatters on an atom at energy-momentum transfer $q = (T, \vec{q})$. In what follows the recoil of atoms is neglected because of the reasonable assumption $T \gg 2E_\nu/M$, M is the nuclear mass.

The atomic target is supposed to be unpolarized and in its ground state $|0\rangle$ with the corresponding energy E_0 . It is also supposed that $T \ll m_e$ and $\alpha Z \ll 1$, where Z is the nuclear charge and α is the fine-structure constant, so that the initial and final electronic systems can be treated nonrelativistically. The neutrino states are described by the Dirac spinors assuming $m_\nu = 0$.

In the considered low-energy limit the neutrino magnetic moment contribution to the electromagnetic vertex (3.18) can be expressed in the following form

$$\Lambda_\mu = \frac{\mu_\nu}{2m_e} \sigma_{\mu\nu} q^\nu. \quad (D1)$$

Thus the magnetic moment interaction of a neutrino with the atomic electrons is described by the Lagrangian

$$L_{int} = \mu_\nu \bar{\psi}(k') \sigma_{\alpha\beta} \psi_j(k) q^\alpha A^\beta, \quad (D2)$$

where the electromagnetic field $A_\mu = (A_0, \vec{A})$ of the atomic electrons is $A_0(\vec{q}) = \sqrt{4\pi\alpha} \rho(\vec{q})/\vec{q}^2$, $\vec{A}(\vec{q}) = \sqrt{4\pi\alpha} \vec{j}(\vec{q})/\vec{q}^2$, where $\rho(\vec{q})$ and $\vec{j}(\vec{q})$ are the Fourier transforms of the electron number density and current density operators, respectively,

$$\rho(\vec{q}) = \sum_{a=1}^Z \exp(i\vec{q} \cdot \vec{r}_a), \quad (D3)$$

$$\vec{j}(\vec{q}) = -\frac{i}{2m} \sum_{a=1}^Z \left[\exp(i\vec{q} \cdot \vec{r}_a) \frac{\partial}{\partial \vec{r}_a} + \frac{\partial}{\partial \vec{r}_a} \exp(i\vec{q} \cdot \vec{r}_a) \right], \quad (D4)$$

and the sums run over the positions \vec{r}_a of all the Z electrons in the atom. The double differential cross section can be presented as

$$\frac{d^2\sigma_{(\mu)}}{dT d\vec{q}^2} = \left(\frac{d^2\sigma_{(\mu)}}{dT d\vec{q}^2} \right)_{\parallel} + \left(\frac{d^2\sigma_{(\mu)}}{dT d\vec{q}^2} \right)_{\perp}, \quad (D5)$$

where

$$\frac{d^2\sigma_{(\mu)}}{dT d\vec{q}^2_{\parallel}} = 4\pi\alpha \frac{\mathbb{P}_{\nu}^2}{\vec{q}^2} \cdot \left(1 - \frac{T^2}{\vec{q}^2}\right) S(T, \vec{q}^2), \quad (\text{D6})$$

and

$$\frac{d^2\sigma_{(\mu)}}{dT d\vec{q}^2_{\perp}} = 4\pi\alpha \frac{\mathbb{P}_{\nu}^2}{\vec{q}^2} \left(1 - \frac{\vec{q}^2}{4E_{\nu}^2}\right) R(T, \vec{q}^2), \quad (\text{D7})$$

where $S(T, \vec{q}^2)$, also known as the dynamical structure factor (Fano, 1963), and $R(T, \vec{q}^2)$ are

$$S(T, \vec{q}^2) = \sum_n \delta(T - E_n + E_0) |\langle n | \rho(\vec{q}) | 0 \rangle|^2, \quad (\text{D8})$$

$$R(T, \vec{q}^2) = \sum_n \delta(T - E_n + E_0) |\langle n | j_{\perp}(\vec{q}) | 0 \rangle|^2, \quad (\text{D9})$$

with j_{\perp} being the \vec{j} component perpendicular to \vec{q} and parallel to the scattering plane, which is formed by the incident and final neutrino momenta. The sums in Eqs. (D8) and (D9) run over all the states $|n\rangle$ with energies E_n of the electron system, with $|0\rangle$ being the initial state.

The longitudinal term (D6) is associated with atomic excitations induced by the force that the neutrino magnetic moment exerts on electrons in the direction parallel to \vec{q} . The transverse term (D7) corresponds to the exchange of a virtual photon which is polarized as a real one, that is, perpendicular to \vec{q} . It resembles a photoabsorption process when $\vec{q}^2 \rightarrow T$ and the virtual-photon four-momentum thus approaches a physical value, $q^2 \rightarrow 0$. Due to selection rules, the longitudinal and transverse excitations do not interfere (see Fano (1963) for detail).

The factors $S(T, \vec{q}^2)$ and $R(T, \vec{q}^2)$ are related to respectively the density-density $F(T, \vec{q}^2)$ and current-current $L(T, \vec{q}^2)$ Green's functions

$$S(T, \vec{q}^2) = \frac{1}{\pi} \text{Im} F(T, q^2), \quad (\text{D10})$$

$$R(T, \vec{q}^2) = \frac{1}{\pi} \text{Im} L(T, \vec{q}^2), \quad (\text{D11})$$

where

$$\begin{aligned} F(T, \vec{q}^2) &= \sum_n \frac{|\langle n | \rho(\vec{q}) | 0 \rangle|^2}{T - E_n + E_0 - i\epsilon} \\ &= \left\langle 0 \left| \rho(-\vec{q}) \frac{1}{T - H + E_0 - i\epsilon} \rho(\vec{q}) \right| 0 \right\rangle, \end{aligned} \quad (\text{D12})$$

$$\begin{aligned} L(T, \vec{q}^2) &= \sum_n \frac{|\langle n | j_{\perp}(\vec{q}) | 0 \rangle|^2}{T - E_n + E_0 - i\epsilon} \\ &= \left\langle 0 \left| j_{\perp}(-\vec{q}) \frac{1}{T - H + E_0 - i\epsilon} j_{\perp}(\vec{q}) \right| 0 \right\rangle, \end{aligned} \quad (\text{D13})$$

H being the Hamiltonian for the system of electrons. For small values of \vec{q} , in particular, such that $|\vec{q}| \sim T$, only the lowest-order non-zero terms of the expansion of Eqs. (D10) and (D11) in powers of \vec{q}^2 are of relevance (the so-called dipole approximation). In this case, one has (Kouzakov and Studenikin, 2011b)

$$R(T, \vec{q}^2) = \frac{T^2}{\vec{q}^2} S(T, \vec{q}^2). \quad (\text{D14})$$

Note that this ratio is much smaller than unity practically for all \vec{q}^2 values involved in Eqs. (D6) and (D7). Thus, taking into account the foregoing arguments, one might expect the transverse component to play a minor role in Eq. (D5). The authors of Wong *et al.* (2010), however, came to the contrary conclusion that this component dramatically enhances due to atomic ionization when $T \sim \varepsilon_b$. The enhancement mechanism proposed in Wong *et al.* (2010) is based on an analogy with the photoionization process. As mentioned above, when $\vec{q}^2 \rightarrow T^2$ the virtual-photon momentum approaches the physical regime $q^2 = 0$. In this case, we have for the integrand in Eq. (D7)

$$\left. \frac{R(T, \vec{q}^2)}{\vec{q}^2} \right|_{\vec{q}^2 \rightarrow T^2} = \frac{\sigma_{\gamma}(T)}{4\pi^2 \alpha T}, \quad (\text{D15})$$

where $\sigma_{\gamma}(T)$ is the photoionization cross section at the photon energy T (Akhiezer and Berestetsky, 1965). The limiting form (D15) was used in Wong *et al.* (2010) in the whole integration interval. Such a procedure is obviously incorrect, for the integrand rapidly falls down as \vec{q}^2 ranges from T^2 up to $4E_{\nu}^2$, especially when $\vec{q}^2 \gtrsim r_a^{-2}$, where r_a is a characteristic atomic size (within the Thomas-Fermi model $r_a^{-1} = Z^{1/3} \alpha m_e$ (Landau and Lifshitz, 1977)). This fact reflects a strong departure from the real-photon regime. For this reason we can classify the enhancement of the DCS determined in Wong *et al.* (2010) as spurious.

Taking into account Eq. (D14), the experimentally measured single-differential inclusive cross section is, to a good approximation, given by (see e.g. in Voloshin (2010); Kouzakov and Studenikin (2011b); Kouzakov *et al.* (2011a))

$$\frac{d\sigma_{(\mu)}}{dT} = 4\pi\alpha \frac{\mathbb{P}_{\nu}^2}{\vec{q}^2} \cdot \int_{T^2}^{4E_{\nu}^2} S(T, \vec{q}^2) \frac{d\vec{q}^2}{\vec{q}^2}. \quad (\text{D16})$$

The standard electroweak contribution to the cross section can be similarly expressed in terms of the same factor $S(T, \vec{q}^2)$ (Voloshin, 2010) as

$$\begin{aligned} \frac{d\sigma_{EW}}{dT} &= \frac{G_F^2}{4\pi} (1 + 4 \cdot \sin^2 \theta_W + 8 \sin^4 \theta_W) \\ &\times \int_{T^2}^{4E_{\nu}^2} S(T, \vec{q}^2) d\vec{q}^2, \end{aligned} \quad (\text{D17})$$

where the factor $S(T, \vec{q}^2)$ is integrated over \vec{q}^2 with a unit weight, rather than \vec{q}^{-2} as in Eq.(D16).

The kinematical limits for \vec{q}^2 in an actual neutrino scattering are explicitly indicated in Eqs.(D16) and (D17). At large E_ν , typical for the reactor neutrinos, the upper limit can in fact be extended to infinity, since in the discussed here nonrelativistic limit the range of momenta $\sim E_\nu$ is indistinguishable from infinity. The lower limit can be shifted to $\vec{q}^2 = 0$, since the contribution of the region of $\vec{q}^2 < T^2$ can be expressed in terms of the photoelectric cross section (Voloshin, 2010) and is negligibly small (at the level of below one percent in the considered range of T). For this reason we henceforth discuss the momentum-transfer integrals in Eqs. (D16) and (D17) running from $\vec{q}^2 = 0$ to $\vec{q}^2 = \infty$:

$$I_1(T) = \int_0^\infty S(T, \vec{q}^2) \frac{d\vec{q}^2}{\vec{q}^2}, \quad (\text{D18})$$

and

$$I_2(T) = \int_0^\infty S(T, \vec{q}^2) d\vec{q}^2. \quad (\text{D19})$$

For a free electron, which is initially at rest, the density-density correlator is the free particle Green's function

$$F_{(FE)}(T, \vec{q}^2) = \left(T - \frac{\vec{q}^2}{2m} - i\epsilon \right)^{-1} \quad (\text{D20})$$

so that the dynamical structure factor is given by $S_{(FE)}(T, \vec{q}^2) = \delta(T - \vec{q}^2/2m)$, and the discussed here integrals are in the free-electron limit as follows:

$$I_1^{(FE)} = \int_0^\infty S_{(FE)}(T, \vec{q}^2) \frac{d\vec{q}^2}{\vec{q}^2} = \frac{1}{T}, \quad (\text{D21})$$

$$I_2^{(FE)} = \int_0^\infty S_{(FE)}(T, \vec{q}^2) d\vec{q}^2 = 2m. \quad (\text{D22})$$

Clearly, these expressions, when used in the formulas (D16) and (D17), result in the free-electron cross sections,

$$\frac{d\sigma_{(\mu)}}{dT} = 4\pi \alpha \mathbb{P}_\nu^2 \cdot \left(\frac{1}{T} - \frac{1}{E_\nu} \right) \quad (\text{D23})$$

and

$$\begin{aligned} \frac{d\sigma_{EW}}{dT} &= \frac{G_F^2 m}{2\pi} (1 + 4 \sin^2 \theta_W + 8 \sin^4 \theta_W) \\ &\times \left[1 + O\left(\frac{T}{E_\nu}\right) \right], \end{aligned} \quad (\text{D24})$$

correspondingly.

Now we consider neutrino scattering on an electron bound in an atom following consideration of Kouzakov *et al.* (2011a). The binding effects generally deform the

density-density Green's function, so that both the integrals (D18) and (D19) are somewhat modified. Namely, the binding effects spread the free-electron δ -peak in the dynamical structure function at $\vec{q}^2 = 2mT$ and also shift it by the scale of characteristic electron momenta in the bound state.

We consider the scattering on just one electron. The Hamiltonian for the electron has the form $H = p^2/2m + V(r)$, and the density-density Green's function from Eq.(D12) can be written as

$$\begin{aligned} F(T, \vec{q}^2) &= \left\langle 0 \left| e^{-i\vec{q}\cdot\vec{r}} [T - H(\vec{p}, \vec{r}) + E_0]^{-1} e^{i\vec{q}\cdot\vec{r}} \right| 0 \right\rangle \\ &= \left\langle 0 \left| [T - H(\vec{p} + \vec{q}, \vec{r}) + E_0]^{-1} \right| 0 \right\rangle \\ &= \left\langle 0 \left| \left[T - \frac{\vec{q}^2}{2m} - \frac{\vec{p}\cdot\vec{q}}{m} - H(\vec{p}, \vec{r}) + E_0 \right]^{-1} \right| 0 \right\rangle, \end{aligned} \quad (\text{D25})$$

where the infinitesimal shift $T \rightarrow T - i\epsilon$ is implied.

Clearly, a nontrivial behavior of the latter expression in Eq.(D25) is generated by the presence of the operator $(\vec{p}\cdot\vec{q})$ in the denominator, and the fact that it does not commute with the Hamiltonian H . Thus an analytical calculation of the Green's function as well as the dynamical structure factor is feasible in only few specific problems. In Kouzakov *et al.* (2011a) we present such a calculation for ionization from the $1s$, $2s$, and $2p$ hydrogen-like states. In particular, we find that the deviation of the discussed integrals (D18) and (D19) from their free values are very small: the largest deviation is exactly at the ionization threshold, where, for instance, each of the $1s$ integrals is equal to the free-electron value multiplied by the factor $(1 - 7e^{-4}/3) \approx 0.957$ ¹⁴. The same conclusion was also obtained in Kouzakov and Studenikin (2011b) where the $1s$ case was examined numerically.

The problem of calculating the integrals (D18) and (D19) however can be solved in the semiclassical limit, where one can neglect the noncommutativity of the momentum \vec{p} with the Hamiltonian, and rather treat this operator as a number vector. Taking also into account that $(H - E_0)|0\rangle = 0$, one can then readily average the latter expression in Eq.(D25) over the directions of \vec{q}^2 and find the formula for the dynamical structure factor:

$$\begin{aligned} S(T, \vec{q}^2) &= \frac{m}{2|\vec{p}||\vec{q}|} \left[\theta \left(T - \frac{\vec{q}^2}{2m} + \frac{|\vec{p}||\vec{q}|}{m} \right) \right. \\ &\quad \left. - \theta \left(T - \frac{\vec{q}^2}{2m} - \frac{|\vec{p}||\vec{q}|}{m} \right) \right], \end{aligned} \quad (\text{D26})$$

¹⁴ It can be also noted that both integrals are modified in exactly the same proportion, so that their ratio is not affected at any T : $I_2(T)/I_1(T) = 2mT$. We find however that this exact proportionality is specific for the ionization from the ground state in the Coulomb potential.

where θ is the standard Heaviside step function. The expression in Eq.(D26) is nonzero only in the range of $|\vec{q}|$ satisfying the condition $-|\vec{p}||\vec{q}|/m < T - |\vec{q}|^2/2m < |\vec{p}||\vec{q}|/m$, i.e. between the (positive) roots of the binomials in the arguments of the step functions: $\vec{q}_{min}^2 = \sqrt{2mT + |\vec{p}|^2} - |\vec{p}|$ and $\vec{q}_{max}^2 = \sqrt{2mT + |\vec{p}|^2} + |\vec{p}|$. One can notice that the previously mentioned ‘spread and shift’ of the peak in the dynamical structure function in this limit corresponds to a flat pedestal between \vec{q}_{min}^2 and \vec{q}_{max}^2 . The calculation of the integrals (D18) and (D19) with the expression (D26) is straightforward, and yields the free-electron expressions (D21) and (D22) for the discussed here integrals in the semiclassical (WKB) limit¹⁵:

$$I_1^{(WKB)} = \frac{1}{T}, \quad I_2^{(WKB)} = 2m. \quad (D27)$$

The difference from the pure free-electron case however is in the range of the energy transfer T . Namely, the expressions (D27) are applicable in this case only above the ionization threshold, i.e. at $T \geq |E_0|$. Below the threshold the electron becomes ‘inactive’.

Appendix E: Calculation of potentials

In this Appendix we describe the calculation of the potentials in Eqs. (6.11), (6.13), (6.14) and (7.35). It is convenient to start by writing the potential in Eq. (6.7) (for $q = 0$ and $j^\mu = 0$) as

$$V_{h_i \rightarrow h_f} = \frac{1}{4E} \text{Tr} \left[u^{(h_i)}(p) \overline{u^{(h_f)}(p)} \sigma_{\mu\nu} F^{\mu\nu} (\mathbb{I} + i\epsilon\gamma_5) \right]. \quad (E1)$$

For the helicity-conserving potential $V_{h \rightarrow h}$, we have

$$\frac{u^{(h)}(p) \overline{u^{(h)}(p)}}{2m} = \Lambda_+(p) P_h, \quad (E2)$$

with the energy and helicity projection operators $\Lambda_+(p)$ and P_h given in Eqs. (A46) and (A48). Using the values of the traces of products of γ matrices given in Eqs. (A37)–(A39), we obtain

$$V_{h \rightarrow h} = -\frac{h}{2E} [\mathbb{I} \epsilon_{\alpha\beta\mu\nu} p^\alpha s^\beta F^{\mu\nu} - 2\epsilon F^{\mu\nu} s_\mu p_\nu]. \quad (E3)$$

Then, taking into account the expressions in Eq. (A72) for the electric and magnetic fields, we obtain Eq. (6.11).

In order to calculate the helicity-flipping potential $V_{-h \rightarrow h}$, we define the helicity-flipping matrix

$$F = \vec{\tau} \cdot \vec{\gamma} \gamma_5, \quad (E4)$$

where $\vec{\tau}$ is an arbitrary unit vector orthogonal to \vec{p} , i.e. such that

$$|\vec{\tau}|^2 = 1, \quad \vec{\tau} \cdot \vec{p} = 0. \quad (E5)$$

One can check that

$$[F, \not{p}] = \{F, \gamma_5\} = \{F, \gamma_5^5\} = 0, \quad (E6)$$

and

$$F^2 = 1, \quad \gamma^0 F^\dagger \gamma^0 = F, \quad F P_h = P_{-h} F. \quad (E7)$$

Therefore, we have

$$u^{(-h)}(p) = F u^{(h)}(p), \quad (E8)$$

and

$$\frac{u^{(-h)}(p) \overline{u^{(h)}(p)}}{2m} = F \Lambda_+(p) P_h = P_{-h} \Lambda_+(p) F. \quad (E9)$$

Plugging the expression (E9) in Eq. (E1) for $h = h_f = -h_i$ and using the values of the traces of products of γ matrices given in Eqs. (A37)–(A39), we obtain

$$V_{-h \rightarrow h} = -\frac{\tau^k}{2E} \left[\mathbb{I} (\epsilon^{k\alpha\mu\nu} p_\alpha F_{\mu\nu} + 2imh F^{k\alpha} s_\alpha) + \epsilon (2F^{k\alpha} p_\alpha - imh \epsilon^{k\alpha\mu\nu} s_\alpha F_{\mu\nu}) \right]. \quad (E10)$$

Note that the ih factors are correct in order to satisfy the hermiticity constraint in Eq. (6.2).

Considering the expressions in Eq. (A72) for the electric and magnetic fields, one can find

$$V_{-h \rightarrow h} = \mathbb{I} \left(-\vec{\tau} \cdot \vec{B} - ih\vec{\tau} \cdot \frac{\vec{p} \times \vec{B}}{|\vec{p}|} + \vec{\tau} \cdot \frac{\vec{p} \times \vec{E}}{E} - ih\frac{|\vec{p}|}{E} \vec{\tau} \cdot \vec{E} \right) + \epsilon \left(-\vec{\tau} \cdot \vec{E} - ih\vec{\tau} \cdot \frac{\vec{p} \times \vec{E}}{|\vec{p}|} - \vec{\tau} \cdot \frac{\vec{p} \times \vec{B}}{E} + ih\frac{|\vec{p}|}{E} \vec{\tau} \cdot \vec{B} \right). \quad (E11)$$

Therefore, only the components of the electric and magnetic fields orthogonal to \vec{p} contribute to the helicity-flipping potential.

If we have only an electric or a magnetic field, choosing $\vec{\tau}$ antiparallel to the component of the electric or magnetic field orthogonal to \vec{p} , we obtain the helicity-flipping potentials in Eqs. (6.13) and (6.14). In the general case of an electric and a magnetic field which are not parallel, one must use the general equation (E11), which can be conveniently expressed in terms of the fields components. Choosing, for example,

$$\vec{p} = (0, 0, |\vec{p}|), \quad \vec{\tau} = (-1, 0, 0), \quad (E12)$$

¹⁵ The appearance of the free-electron expressions here is not surprising, since the equation (D26) can be also viewed as the one for scattering on an electron boosted to the momentum p

we obtain

$$V_{-h \rightarrow h} = \mu \left(B^1 - ihB^2 + \frac{|\vec{p}|}{E} E^2 + ih \frac{|\vec{p}|}{E} E^1 \right) + \epsilon \left(E^1 - ihE^2 - \frac{|\vec{p}|}{E} B^2 - ih \frac{|\vec{p}|}{E} B^1 \right), \quad (\text{E13})$$

with $\vec{E} = (E^1, E^2, E^3)$ and $\vec{B} = (B^1, B^2, B^3)$. The arbitrariness introduced by the choice of $\vec{\tau}$ is only apparent, because the phase of $V_{-h \rightarrow h}$ does not have physical effects. For the physical absolute value of the potential we find

$$|V_{-h \rightarrow h}|^2 = \mu^2 \left(B_\perp^2 + \frac{|\vec{p}|}{E} E_\perp^2 + 2 \frac{\vec{p} \cdot \vec{B}_\perp \times \vec{E}_\perp}{E} \right) + \epsilon^2 \left(E_\perp^2 + \frac{|\vec{p}|}{E} B_\perp^2 - 2 \frac{\vec{p} \cdot \vec{E}_\perp \times \vec{B}_\perp}{E} \right) + 2\mu\epsilon \frac{m}{E} \vec{B}_\perp \cdot \vec{E}_\perp, \quad (\text{E14})$$

with $\vec{E}_\perp = (0, E^2, E^3)$ and $\vec{B}_\perp = (0, B^2, B^3)$. Hence, it is clear that $|V_{-h \rightarrow h}|$ does not depend on the choice of $\vec{\tau}$.

The dependence of the contribution of the magnetic field on ϵ and that of the electric field on μ are a consequence of the relativistic transformation of the electric and magnetic fields and the fact that the classical electric and magnetic dipole moments are defined for a non-relativistic particle through Eqs. (6.9) and (6.10), which establish the behavior of the non-relativistic particle in electric and magnetic fields. In fact, for a non-relativistic neutrino ($|\vec{p}| \ll E$) we have

$$V_{-h \rightarrow h}^{\text{nr}} \simeq \mu (B_{\text{rf}}^1 - ihB_{\text{rf}}^2) + \epsilon (E_{\text{rf}}^1 - ihE_{\text{rf}}^2), \quad (\text{E15})$$

where the index “rf” indicates the rest frame of the neutrino, with the helicity defined as the projection of twice the spin on the z -axis. One can see that for a non-relativistic neutrino the contribution of the magnetic field depends only on the magnetic dipole moment μ and the contribution of the electric field depends only on the electric dipole moment ϵ . Considering now a frame in which the neutrino is relativistic, the components of the electric and magnetic fields are given by

$$E^1 = \gamma (E_{\text{rf}}^1 + vB_{\text{rf}}^2), \quad (\text{E16})$$

$$E^2 = \gamma (E_{\text{rf}}^2 - vB_{\text{rf}}^1), \quad (\text{E17})$$

$$E^3 = E_{\text{rf}}^3, \quad (\text{E18})$$

$$B^1 = \gamma (B_{\text{rf}}^1 - vE_{\text{rf}}^2), \quad (\text{E19})$$

$$B^2 = \gamma (B_{\text{rf}}^2 + vE_{\text{rf}}^1), \quad (\text{E20})$$

$$B^3 = B_{\text{rf}}^3, \quad (\text{E21})$$

with $v = |\vec{p}|/E$ and $\gamma = (1 - v^2)^{-1/2} = E/m$. Plugging these components of \vec{E} and \vec{B} in Eq. (E13), we obtain

$$V_{-h \rightarrow h} = \mu \frac{m}{E} (B_{\text{rf}}^1 - ihB_{\text{rf}}^2) + \epsilon \frac{m}{E} (E_{\text{rf}}^1 - ihE_{\text{rf}}^2), \quad (\text{E22})$$

Therefore, the contribution of the magnetic field in the rest frame depends only on the magnetic dipole moment μ and the contribution of the electric field in the rest depends only on the electric dipole moment ϵ . If, for example, there is no electric field in the rest frame, $V_{-h \rightarrow h}$ depends only on the magnetic dipole moment μ . In this case the coefficient of ϵ in Eq. (E13) vanishes because the components of \vec{E} and \vec{B} are given by the relativistic transformations (E16)–(E21) with $\vec{E}_{\text{rf}} = 0$ and the contribution of \vec{E} in the coefficient of μ is due to the same relativistic transformations.

Let us consider finally the contribution of the anapole moment to the helicity-flipping potential. From the last term in Eq. (6.7), using Eq. (E9) we obtain

$$V_{-h \rightarrow h} = \mathfrak{a} \frac{m}{E} \vec{j} \cdot \vec{\tau} + i \mathfrak{a} h \epsilon^{\alpha\beta\mu\kappa} \frac{p_\alpha}{E} s_\beta j_\mu \tau^\kappa. \quad (\text{E23})$$

Taking into account that \vec{s} and \vec{p} are parallel (see Eq. (A49)), we can write this expression as

$$V_{-h \rightarrow h} = \mathfrak{a} \frac{m}{E} \vec{j} \cdot \vec{\tau} + i \frac{m}{E} \vec{a} \cdot \vec{j} \times \vec{\tau}. \quad (\text{E24})$$

Then, choosing $\vec{\tau}$ along the component of \vec{j} orthogonal to \vec{a} (which is defined in Eq. (7.34) as parallel to \vec{p}), the second term vanishes and we obtain Eq. (7.35).

Appendix F: Quasiclassical spin evolution in external fields

In this Appendix we show how the neutrino spin evolution can be described in general case when the neutrino is subjected to general types of non-derivative interactions with external fields (Dvornikov and Studenikin, 2002) (see also Bergmann *et al.* (1999)). Let the neutrino interactions are given by the Lagrangian

$$-\mathcal{L} = g_s s(x) \bar{\nu} \nu + g_p \pi(x) \bar{\nu} \gamma^5 \nu + g_v V^\mu(x) \bar{\nu} \gamma_\mu \nu + g_a A^\mu(x) \bar{\nu} \gamma_\mu \gamma^5 \nu + \frac{g_t}{2} T^{\mu\nu} \bar{\nu} \sigma_{\mu\nu} \nu + \frac{g'_t}{2} \Pi^{\mu\nu} \bar{\nu} \sigma_{\mu\nu} \gamma^5 \nu, \quad (\text{F1})$$

where $s, \pi, V^\mu = (V^0, \vec{V})$, $A^\mu = (A^0, \vec{A})$, $T_{\mu\nu} = (\vec{a}, \vec{b})$, $\Pi_{\mu\nu} = (\vec{c}, \vec{d})$ are scalar, pseudoscalar, vector, axial-vector, tensor and pseudotensor fields, respectively. This Lagrangian accounts for a wide set of non-derivative neutrino interactions with external fields. We introduce the neutrino spin operator in a usual form,

$$\vec{O} = \gamma_0 \vec{\Sigma} - \gamma_5 \frac{\vec{p}}{p_0} - \gamma_0 \frac{\vec{p}(\vec{p} \cdot \vec{\Sigma})}{p_0^2(p_0^2 + m_\nu^2)}, \quad (\text{F2})$$

where $\vec{\Sigma} = \gamma^0 \gamma^5 \vec{\gamma}$. Its overage over the neutrino stationary states gives the neutrino three-dimensional spin vector

$$\langle \vec{O} \rangle = \vec{\zeta}_\nu \quad (\text{F3})$$

that determines the four-dimensional spin vector given by Eq. (6.86) in Section VI. The corresponding spin evolution equation is obtained in Dvornikov and Studenikin (2002),

$$\begin{aligned} \frac{d\vec{\zeta}_\nu}{dt} = & 2g_a \left\{ A^0 [\vec{\zeta}_\nu \times \vec{\beta}] - \frac{(\vec{A}\vec{\beta})[\vec{\zeta}_\nu \times \vec{\beta}]}{1 + \gamma^{-1}} - \frac{1}{\gamma} [\vec{\zeta}_\nu \times \vec{A}] \right\} \\ & + 2g_t \left\{ [\vec{\zeta}_\nu \times \vec{b}] - \frac{(\vec{\beta}\vec{b})[\vec{\zeta}_\nu \times \vec{\beta}]}{1 + \gamma^{-1}} + [\vec{\zeta}_\nu \times [\vec{a} \times \vec{\beta}]] \right\} \\ & + 2ig'_t \left\{ [\vec{\zeta}_\nu \times \vec{c}] - \frac{(\vec{\beta}\vec{c})[\vec{\zeta}_\nu \times \vec{\beta}]}{1 + \gamma^{-1}} - [\vec{\zeta}_\nu \times [\vec{d} \times \vec{\beta}]] \right\}, \end{aligned} \quad (\text{F4})$$

where $\gamma = p_\nu^0/m_\nu$ and $\vec{\beta}$ is the neutrino speed. This is a rather general equation for the neutrino spin evolution that can be also used for the description of neutrino spin oscillations in different environments, such as moving and polarized matter with external electromagnetic fields (see Studenikin (2004, 2007)). The $SL\nu$ in gravitational fields has been studied (see Grigoriev *et al.* (2005a)) on the basis of a neutrino spin evolution equation (F4).

The Lorentz invariant form of Eq. (F4) can be obtained using the four-dimensional spin vector S^μ which is determined by the three-dimensional spin vector $\vec{\zeta}_\nu$ in accordance with the relation:

$$S^\mu = \left(\frac{(\vec{\zeta}_\nu \vec{p})}{m_\nu}, \vec{\zeta}_\nu + \frac{\vec{p}(\vec{\zeta}_\nu \vec{p})}{m_\nu(m_\nu + p_\nu^0)} \right). \quad (\text{F5})$$

Thus, we get the Lorentz invariant form for the neutrino spin S^μ evolution equation accounting for the general interactions with external fields

$$\begin{aligned} \frac{dS^\mu}{d\tau} = & 2g_t(T^{\mu\nu}S_\nu - u^\mu T^{\lambda\rho}u_\lambda S_\rho) + \\ & 2ig'_t(\tilde{\Pi}^{\mu\nu}S_\nu - u^\mu \tilde{\Pi}^{\lambda\rho}u_\lambda S_\rho) + 2g_a G^{\mu\nu}S_\nu, \end{aligned} \quad (\text{F6})$$

where $G^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta}A_\alpha u_\beta$, $u^\mu = (1, \vec{\beta})E_\nu/m_\nu$, $\tilde{\Pi}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta}\Pi_{\alpha\beta}/2$. The tensor $G_{\mu\nu}$ can be expressed through two vectors $G_{\mu\nu} = (-\vec{P}, \vec{M})$ which are presented in the form,

$$\vec{M} = \gamma(A^0\vec{\beta} - \vec{A}), \quad \vec{P} = -\gamma[\vec{\beta} \times \vec{A}]. \quad (\text{F7})$$

The derivation in the left-handed side of this equation is taken over the neutrino proper time $\tau = \gamma^{-1}$, where t is the time in the laboratory frame of reference.

Note that the obtained Eq. (F6) can be considered as the generalized Bargmann-Michel-Telegdi equation and it was used in Egorov *et al.* (2000); Lobanov and Studenikin (2001) for description of the neutrino spin and flavor oscillations in arbitrary electromagnetic fields. Some general aspects of the neutrino spin dynamics in case of non-minimal couplings with an external magnetic field was studied in Bernardini (2006).

Appendix G: Neutrino wave functions in magnetic field and matter

In this Appendix we derive exact solutions of the Dirac equation for two cases: 1) for an electron in a constant magnetic field and 2) for a neutrino in presence of matter. The electron wave function in magnetic field is used in Section VI, in calculations a neutrino mass-operator, of the beta decay of a neutron and $\nu\bar{\nu}$ synchrotron radiation by an electron in magnetic field. The neutrino wave function in matter is used in Subsection V.E in studies of the spin light of neutrino in matter.

Following Balantsev and Studenikin (2012) and Balantsev *et al.* (2011), we derive two exact solutions for the Dirac equation for two considered cases starting with general equation for a charged particle wave functions that accounts both for the presence of a magnetic field and matter. In the case of the standard model interaction of an electron neutrino and electron with matter composed of neutrons, the modified Dirac equations as is given by Studenikin and Ternov (2005), Studenikin (2011) and Studenikin (2008)

$$\left\{ i\gamma_\mu(\partial^\mu + \mathbf{q}_l A^\mu) - \frac{1}{2}\gamma_\mu(c_l + \gamma_5)\tilde{f}^\mu - m_l \right\} \Psi^{(l)}(\vec{r}, t) = 0, \quad (\text{G1})$$

where for the case of the electron $m_l = m_e$, $c_l = c_e = 1 - 4\sin^2\theta_W$ and $\mathbf{q}_l = -e$. For neutrinos $m_l = m_\nu$, $c_l = c_\nu = 1$ and $\mathbf{q}_l = \mathbf{q}_\nu$ is the possible neutrino millicharge (Balantsev and Studenikin, 2012) (see Subsection VII.A). For unpolarized and not moving matter $\tilde{f}^\mu = G(n, 0, 0, 0)$, n is the matter number density, $G = \frac{G_F}{\sqrt{2}}$, and for the magnetic field potential $A^\mu = (0, 0, Bx, 0)$.

Equation (G1) can be rewritten in the Hamiltonian form

$$i\frac{\partial}{\partial t}\Psi(\vec{r}, t) = \hat{H}\Psi(\vec{r}, t), \quad (\text{G2})$$

$$\hat{H} = \gamma^0\vec{\gamma}(\vec{p} + \mathbf{q}_l\vec{A}) + m_l\gamma^0 + \frac{Gn}{2}(c_l + \gamma^5). \quad (\text{G3})$$

The solution of Eq. (G2) due to symmetries can be sought in the form

$$\Psi^{(l)}(\vec{r}, t) = e^{-ip_0t + ip_2y + ip_3z} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}. \quad (\text{G4})$$

Substituting (G4) into (G2) and introducing the increasing and decreasing operators,

$$\hat{a} = \frac{1}{\sqrt{2}}(\eta + \frac{d}{d\eta}), \quad \hat{a}^+ = \frac{1}{\sqrt{2}}(\eta - \frac{d}{d\eta}), \quad (\text{G5})$$

where

$$\eta = x\sqrt{\gamma_l} + \frac{p_2}{\sqrt{\gamma_l}}, \quad \gamma_l = \mathbf{q}_l B, \quad (\text{G6})$$

we arrive at a system of linear equations for the particle wave function components:

$$\begin{aligned} (\tilde{p}_0 - m_l)\psi_1 + i\sqrt{2q_l B}\hat{a}\psi_4 - \left(p_3 - \frac{Gn}{2}\right)\psi_3 &= 0, \\ (\tilde{p}_0 - m_l)\psi_2 - i\sqrt{2q_l B}\hat{a}^+\psi_3 + \left(p_3 + \frac{Gn}{2}\right)\psi_4 &= 0, \\ (\tilde{p}_0 + m_l)\psi_3 + i\sqrt{2q_l B}\hat{a}\psi_2 - \left(p_3 - \frac{Gn}{2}\right)\psi_1 &= 0, \\ (\tilde{p}_0 + m_l)\psi_4 - i\sqrt{2q_l B}\hat{a}^+\psi_1 + \left(p_3 + \frac{Gn}{2}\right)\psi_2 &= 0, \end{aligned} \quad (G7)$$

where $\tilde{p}_0 = p_0 + Gnc_l/2$.

The exact solution for Dirac equation for a particles wave function in presence of the magnetic field and matter can be written in the form

$$\Psi^{(l)}(\vec{r}, t) = \frac{1}{L} e^{-ip_0 t + ip_2 y + ip_3 z} \begin{pmatrix} C_1 U_{N-1}(\eta) \\ iC_2 U_N(\eta) \\ C_3 U_{N-1}(\eta) \\ iC_4 U_N(\eta) \end{pmatrix}. \quad (G8)$$

Then the system of equations for the coefficients C_i , with $i = 1, 2, 3, 4$, is

$$\begin{aligned} (\tilde{p}_0 - m_l)C_1 - \sqrt{2q_l B N}C_4 - \left(p_3 - \frac{Gn}{2}\right)C_3 &= 0, \\ (\tilde{p}_0 - m_l)C_2 - \sqrt{2q_l B N}C_3 + \left(p_3 + \frac{Gn}{2}\right)C_4 &= 0, \\ (\tilde{p}_0 + m_l)C_3 - \sqrt{2q_l B N}C_2 - \left(p_3 - \frac{Gn}{2}\right)C_1 &= 0, \\ (\tilde{p}_0 + m_l)C_4 - \sqrt{2q_l B N}C_1 + \left(p_3 + \frac{Gn}{2}\right)C_2 &= 0. \end{aligned} \quad (G9)$$

From (G9) we get the energy spectrum

$$p_0 = \varepsilon \sqrt{m_l^2 + \left(m_l T^0 - \frac{Gn}{2}\right)^2} + \frac{Gn}{2}c_l, \quad (G10)$$

where

$$T^0 = \frac{s}{m_l} \sqrt{p_3^2 + 2q_l B N}, \quad (G11)$$

$s = \pm 1$, are the eigenvalues of the longitudinal spin polarization operator $\hat{T}^0 = \vec{\sigma} \cdot (\vec{p} + e\vec{A})/m_l$ (Sokolov and Ternov, 1968) and $\varepsilon = \pm 1$ is the sign of the energy. Note that in the presence of the matter potential proportional to Gn the transverse spin polarization operator does not commute with the Hamiltonian (G2), that is a consequence of γ^5 presence in (G1). Taking into account the normalization condition $\sum_i C_i^2 = 1$ the solution of the

system (G9) is

$$\begin{aligned} C_1 &= \frac{1}{2} \sqrt{1 + \frac{m_l}{p_0 - \frac{Gn}{2}c_l}} \sqrt{1 + \frac{p_3}{m_l T^0}}, \\ C_2 &= \frac{s}{2} \sqrt{1 + \frac{m_l}{p_0 - \frac{Gn}{2}c_l}} \sqrt{1 - \frac{p_3}{m_l T^0}}, \\ C_3 &= \frac{s\varepsilon\eta}{2} \sqrt{1 - \frac{m_l}{p_0 - \frac{Gn}{2}c_l}} \sqrt{1 + \frac{p_3}{m_l T^0}}, \\ C_4 &= \frac{\varepsilon\eta}{2} \sqrt{1 - \frac{m_l}{p_0 - \frac{Gn}{2}c_l}} \sqrt{1 - \frac{p_3}{m_l T^0}}, \end{aligned} \quad (G12)$$

where $\eta = \text{sign}(p - s\frac{Gn}{2})$.

From the obtained exact solution of Dirac equation (G1) for a charged massive particle moving in a magnetic field and matter, that is given by Eqs. (G8), (G10), and (G12) it is easy to get the solution for the electron wave function in the magnetic field by “switching off” the matter term $Gn \rightarrow 0$ and the corresponding proper choice for other values: $m_l = m_e$, $c_l = c_e = 1 - 4\sin^2\theta_W$ and $q_l = -e$. In particular, from (G10) one obtains the well known energy spectrum for the electron in a constant magnetic field (see Sokolov and Ternov (1968))

$$p_0 = \sqrt{m_e^2 + p_3^2 + 2eBN} \quad (G13)$$

where $N = 0, 1, 2, \dots$ is the Landau number of the energy levels.

From the obtained general solution for the wave function given by Eqs. (G8), (G10), and (G12) it is also possible to get the wave function for a neutrino moving in matter by “switching off” the magnetic field strength by considering the wave function in the limit $q_\nu B \rightarrow 0$. Of course, the corresponding choice of values, $m_l = m_\nu$, $c_l = c_\nu = 1$ and $q_l = q_\nu$, should be done. When the magnetic field is “switching off”, the maximal number of Landau levels N_{\max} is increasing to infinity, however the product $q_\nu B N = \gamma_\nu N$ remains constant

$$\lim_{\gamma_\nu \rightarrow 0, N \rightarrow \infty} 2\gamma_\nu N = p_\perp^2. \quad (G14)$$

Accounting also the asymptotic behavior of Hermite functions,

$$\lim_{\gamma_\nu \rightarrow 0, N \rightarrow \infty} U_N(\eta) \sim e^{ip_1 x}, \quad (G15)$$

we arrive to the neutrino wave function in matter obtained by Studenikin and Ternov (2005)

$$\Psi_\varepsilon(\vec{r}, t) = \frac{e^{-i(p_0^\varepsilon t - \vec{p}\vec{r})}}{2L^{\frac{3}{2}}} \begin{pmatrix} s \sqrt{1 + \frac{m_\nu}{p_0^\varepsilon - \frac{Gn}{2}}} \sqrt{1 + s\frac{p_3}{p}} e^{i\delta} \\ s\varepsilon\eta \sqrt{1 - \frac{m_\nu}{p_0^\varepsilon - \frac{Gn}{2}}} \sqrt{1 + s\frac{p_3}{p}} \\ s\varepsilon\eta \sqrt{1 - \frac{m_\nu}{p_0^\varepsilon - \frac{Gn}{2}}} \sqrt{1 + s\frac{p_3}{p}} e^{i\delta} \end{pmatrix}, \quad (G16)$$

where $\delta = \arctan p_2/p_1$ and the neutrino energy

$$p_0^\varepsilon = \varepsilon \eta \sqrt{m_\nu^2 + \left(p - s \frac{Gn}{2}\right)^2} + \frac{Gn}{2}. \quad (\text{G17})$$

In the limit of vanishing density of matter, when $n \rightarrow 0$, the wave function (G16) transforms to the vacuum solution of the Dirac equation. The values $s = \pm 1$ specify the two neutrino helicity states, ν_+ and ν_- .

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